Lect 16

Classical

$$\beta(\sigma) = \sum_n \beta_n e^{in\sigma}$$

$$\overline{\beta(\sigma)} = \sum_n \overline{\beta_n} e^{-in\sigma}$$

$$\{\beta_n, \beta_m\} = -\frac{n}{\pi}\delta_{n,-m}$$

$$\{\beta_n, \overline{\beta_m}\} = 0$$

$$\{\overline{\beta_n}, \overline{\beta_m}\} = -\frac{n}{\pi}\delta_{n,-m}$$

$\beta_0, \overline{\beta_0}$ – Casimir

Poisson algebra

$A_0 = \mathbb{C}[\beta_n, \overline{\beta_m}]$

Quantum

$$\hat\beta(\sigma) = \sum_n \hat\beta_n e^{in\sigma}$$

$$\hat{\overline{\beta}}(\sigma) = \sum_n \hat{\overline{\beta}}_n e^{-in\sigma}$$

$$[\hat\beta_n, \hat\beta_m] = i\frac{\hbar}{\pi} n\delta_{n,-m}$$

$$[\hat\beta_n, \hat{\overline{\beta}}_m] = 0$$

$$[\hat{\overline{\beta}}_n, \hat{\overline{\beta}}_m] = i\frac{\hbar}{\pi} n\delta_{n,-m}$$

$\hat\beta_0, \hat{\overline{\beta}}_0 \in \text{center}(A_\hbar)$

$A_\hbar$ = generated by $\hat\beta_n, \hat{\overline{\beta}}_n$

Note: $*: \hat\beta_n \to \hat{\overline{\beta}}_n$, $\hat{\overline{\beta}}_n \to \hat\beta_n$ (like Herm conj.)

extends to a $\mathbb{C}$-antilinear, antiinvolution of $A_\hbar$ $(\bar\hbar = \hbar)$ ($*$-structure) $(ab)^* = b^* a^*, a^{**} = a$

This is why we write $\hat{\overline{\beta}}_n = \hat\beta_n^*$, $\hat{\overline{\beta}}^* = \hat\beta_n^{**} = \beta_n$

indeed: $[\hat\beta_n, \hat\beta_m]^* = -i\frac{\hbar}{\pi} n\delta_{n,-m}$,

$$[\hat\beta_m^*, \hat\beta_n^*] = i\frac{\hbar}{\pi} m\delta_{n,-m} = -\frac{\hbar i}{\pi} n\delta_{n,-m} = [\beta_n, \beta_m]^*$$

Twisted $*$-structure

$(-)^\dagger$

$A_\hbar = B_\hbar \otimes B_\hbar$, $B_\hbar = \langle \hat{\beta}_n \mid [\hat{\beta}_n, \hat{\beta}_m] = i\frac{\hbar n}{\pi}\delta_{n,-m} \rangle$

$\{\hat{\beta}_n\}$ $\{\hat{\beta}_n^*\}$

Claim. $B_\hbar$ has $*$-structure $a \mapsto a^\dagger$

with $\hat{\beta}_n^\dagger = \hat{\beta}_n$

On monomials

$$(\lambda \hat{\beta}_{n_1} \cdots \hat{\beta}_{n_k})^\dagger = \bar{\lambda}\, \hat{\beta}_{n_k} \cdots \hat{\beta}_{n_1}$$

Define the involution $*$ on $A_\hbar$ as

$$(a \otimes b)^* \overset{\text{def}}{=} b^\dagger \otimes a^\dagger$$

in particular

$$(\beta_n \otimes 1)^* = 1 \otimes \beta_n \, , \quad (1 \otimes \beta_n)^* = \beta_n \otimes 1$$

Note: In the notations above $(-)$

$\hat{\beta}_n := \hat{\beta}_n \otimes 1$, $\hat{\beta}_n^* := 1 \otimes \hat{\beta}_n$

Fock space representation of $A_\hbar$

1) Fock Representation of B. $B = \mathbb{C}[\hat{\beta}_0] \otimes_{\mathbb{C}} B'$

B' generated by $\hat{\beta}_n$, $n \neq 0$

$B_+ \subset B'$, B_+ generated by $\hat{\beta}_n$, $n > 0$

Consider 1-dim repr. $\mathcal{L}_-$ on which B_+ acts trivially

$$\hat{\beta}_n \mathcal{L}_- = 0, \quad n > 0$$

Define the representation $\mathcal{F}_-$ on B' as

$$\mathcal{F}_- = \mathrm{Ind}_{B_-}^{B} \mathcal{L}_-$$

Choose a basis in $\mathcal{L}_-$, $\mathcal{L}_- = \mathbb{C} v_-$,

Monomials

$$\hat{\beta}_{-n_1}^{m_1} \cdots \hat{\beta}_{-n_k}^{m_k} v_-$$

form a basis in $\mathcal{F}_-$ with an obviouse action of $\hat{\beta}_n$.

Similarly define repr. $\mathcal{F}_+$ of B' as

$$\mathcal{F}_+ = \mathrm{Ind}_{B_-}^{B} (\mathbb{C} v_+), \quad \hat{\beta}_{-n} v_+ = 0, \quad n > 0$$

Monomials

$$\hat{\beta}_{n_1}^{m_1} \cdots \hat{\beta}_{n_k}^{m_k} v_+$$

form a linear basis in $\mathcal{F}_+$ with an obv. action of B

Theorem. $\exists!$ pairing

$\langle \cdot , \cdot \rangle : \mathcal{F}_+ \otimes \mathcal{F}_- \to \mathbb{C}$ s.t.

- $\langle v_+, v_- \rangle = 1$, $\langle \lambda \ell, x \rangle = \bar{\lambda} \langle \ell, x \rangle$, $\langle \ell, \lambda x \rangle = \lambda \langle \ell, x \rangle$
- $\langle a\ell, x \rangle = \langle \ell, a^+ x \rangle$, $\ell \in \mathcal{F}_+, x \in \mathcal{F}_-$

$\lambda \in \mathbb{C}$, $a \in B'$

Pairing between monomials:

$$\langle \beta_{n_1'}^{m_1'} \cdots \beta_{n_\ell'}^{m_\ell'} v_+ , \beta_{-n_1}^{m_1} \cdots \beta_{-n_\ell}^{m_\ell} v_- \rangle =$$

$$= \delta_{k\ell} \prod_{\alpha=1}^{\ell} \delta_{m_\alpha m_\alpha'} \delta_{n_\alpha n_\alpha'} \left(\frac{i \hbar n_\alpha}{\pi} \right)^{m_\alpha} m_\alpha!$$

$n_\alpha, n_\alpha' > 0$, $m_\alpha, m_\alpha' \geqslant 0$

Now: $\beta_0 \in \mathbb{C}$, define representation $\mathcal{F}_{\beta_0}$ of B as the vector space $\mathcal{F}_+$ with the action of $\hat{\beta}_0$:

$$\hat{\beta}_0 \mathcal{F}_+ = \beta_0 \mathcal{F}_+$$

Similarly $\overline{\mathcal{F}}_{\beta_0}$ as $\mathcal{F}_-$ with

$$\hat{\beta}_0 \overline{\mathcal{F}}_{\beta_0} = \overline{\beta_0}\, \overline{\mathcal{F}}_{\beta_0}$$

We have two representations of B, $\mathcal{F}_{\beta_0}, \overline{\mathcal{F}_{\beta_0}}$

Define

$$H_{\beta_0} = \underbrace{\mathcal{F}_{\beta_0}}_{\mathcal{F}_+} \otimes \underbrace{\overline{\mathcal{F}_{\beta_0}}}_{\mathcal{F}_-}$$

equipped with the bilinear form

$$(\cdot\,,\cdot): H_{\beta_0} \otimes H_{\beta_0} \to \mathbb{C}$$

$$(\underbrace{x_1 \otimes \ell_1}_{\mathcal{F}_+}, \underbrace{x_2 \otimes \ell_2}_{\mathcal{F}_-}) = \langle x_1, \ell_2 \rangle \overline{\langle x_2, \ell_1 \rangle}$$

$$x_i \in \underbrace{\mathcal{F}_{\beta_0}}_{\mathcal{F}_+}, \quad \ell_i \in \underbrace{\overline{\mathcal{F}_{\beta_0}}}_{\mathcal{F}_-}$$

$A_\hbar$-module

Thm This is a Hermian structure on H_{β_0} which is compatible with $*$-involution on $A_\hbar$

- $(\lambda(x_1 \otimes \ell_1), x_2 \otimes \ell_2) = \overline{\lambda}\,(x_1 \otimes \ell_1, x_2 \otimes \ell_2)$

  $(x_1 \otimes \ell_1, \lambda(x_2 \otimes \ell_2)) = \lambda\,(x_1 \otimes \ell_1, x_2 \otimes \ell_2)$

  $\overline{(x_1 \otimes \ell_1, x_2 \otimes \ell_2)} = (x_2 \otimes \ell_2, x_1 \otimes \ell_1)$

- $(x_1 \otimes \ell_1, a x_2 \otimes b \ell_2) = (b^+ x_1 \otimes a^+ \ell_1, x_2 \otimes \ell_2)$

  i.e. $(\alpha, a\beta) = (a^* \beta, \alpha)$, $a \in A_\hbar$, $\alpha, \beta \in H_{\beta_0}$

Proof. H.W.

Normal ordering

Some infinite sums are defined on

$$\hat{\beta}^{\{m\}}_{\{n\}} v_{\beta_0} \overset{\text{def}}{=} \hat{\beta}^{m_1}_{-n_1} \cdots \hat{\beta}^{m_k}_{-n_k} v_{\beta_0}$$

"vacuum"

(∞ sum acts by a finite sum
∞ many terms act by zero)

$x \in \mathcal{F}_{\beta_0}$, $\quad x = \sum_{\substack{\{m\} \\ \{n\}}} x^{\{m\}}_{\{n\}} \hat{\beta}^{\{m\}}_{\{n\}}$ finite sums

$$a = \sum_{\substack{\{m\} \\ \{n\} \\ \text{finite}}} \underset{\mathbb{C}}{a^{\{m\}}_{\{n\}}} \hat{\beta}^{\{m\}}_{\{n\}} \in B_h$$

Assume a is given by an ∞ sum where in monomials $\hat{\beta}^{\{m\}}_{\{n\}}$ all $\hat{\beta}_n$ with $n>0$ are on the right from $\hat{\beta}_n$ with $n<0$

Lemma Such $a \in A_t$ acts on every $x \in \mathcal{F}_{\beta_0}$ as a finite sum.

Def. If a is not of this form (+ on the right, − are on the left) we can reorganize every monomial in a by moving ($+ \to$ right) ($- \to$ to the left)

$\Rightarrow$ $:a:$ = normally ordered a.

<u>Remark</u>. The definition is not "universal" for $B_\hbar$. It depends on representation. For example $: \, :$ for $\mathcal{F}_{\beta_0}$ is opposite to the $: \, :$ for $\overline{\mathcal{F}_{\beta_0}}$.

<u>Quantum fields on S^1</u>

(classical fields on S^1: $\varphi(\sigma), \pi(\sigma) \to \beta(\sigma), \overline{\beta(\sigma)}$)

$$\hat{\beta}(\sigma) = \sum_n \hat{\beta}_n e^{in\sigma}$$

<u>Claim</u>: $[\hat{\beta}(\sigma), \hat{\beta}(\sigma')] = 2\hbar \delta'(\sigma - \sigma')$

<u>Proof</u>. $[\hat{\beta}(\sigma), \hat{\beta}(\sigma')] \overset{\text{distributions}}{=} \sum_{n,m} e^{in\sigma + im\sigma'} [\hat{\beta}_n, \hat{\beta}_m] =$

$$= \sum_{n,m} e^{in\sigma + im\sigma'} \frac{i\hbar n}{\pi} \delta_{n,-m} = \frac{\hbar}{\pi} \sum_n ine^{in(\sigma - \sigma')}$$

$$= 2\hbar \delta'(\sigma - \sigma'), \qquad \delta(\sigma) = \frac{1}{2\pi} \sum_n e^{in\sigma}$$

∎

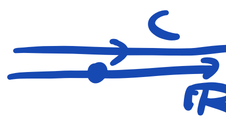
$$\hat\beta(\sigma) = \hat\beta_+(\sigma) + \hat\beta_-(\sigma)$$

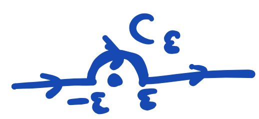
$$\hat\beta_+(\sigma) = \sum_{n>0} \hat\beta_n e^{in\sigma}, \quad \hat\beta_-(\sigma) = \sum_{n\le 0} \hat\beta_n e^{in\sigma}$$

$$\begin{cases} [\hat\beta_+(\sigma), \hat\beta_+(\sigma')] = 0 \\ [\hat\beta_-(\sigma), \hat\beta_-(\sigma')] = 0 \\ [\hat\beta_+(\sigma), \hat\beta_-(\sigma')] = \displaystyle\sum_{n>0} \frac{i\hbar}{\pi} n\, \delta_{n,-m}\, e^{in\sigma + im\sigma'} \\ \quad = -\dfrac{\hbar i}{4\pi} \dfrac{1}{\sin^2\left(\frac{\sigma-\sigma'+i0}{2}\right)}, \end{cases}$$

$\hat\beta_+(\sigma+i0)$

$\hat\beta_-(\sigma-i0)$

$$\int_C f(x) \frac{1}{x+i0}\, dx =$$ 

$\mathbb{R}+i0 =$ 

$$= \int_*^{-\varepsilon} + \int_\varepsilon^* + \int_{C_\varepsilon} f(x) \frac{dx}{x} = \text{p.v.} \int_*^* \frac{f(x)}{x}\, dx - i\pi f(0)$$

$$\int_{C_\varepsilon} f(x)\frac{dx}{x} \xrightarrow{\varepsilon \to 0} f(0)(-\pi i)$$

$$\frac{1}{x+i0} = \text{p.v.} \frac{1}{x} - i\pi \delta(x)$$

Lemma.
$$\frac{1}{\sin^2\left(\frac{\sigma-\sigma'+i0}{2}\right)} = \text{p.v.} \frac{1}{\sin^2\left(\frac{\sigma-\sigma'}{2}\right)} + 4\pi i\, \delta'(\sigma-\sigma')$$

on S^1
$[0, 2\pi]$

Proof. H.W.

$$[\hat{\beta}_+(\sigma), \hat{\beta}_-(\sigma')] = K(\sigma-\sigma') \quad , \quad K(\sigma) = -\frac{i\hbar}{4\pi} \frac{1}{\sh^2\left(\frac{\sigma+i0}{2}\right)}$$

Quantum stress-energy tensor

$$\hat{T}(\sigma) = \frac{1}{4} :\hat{\beta}(\sigma)^2: \quad , \quad \hat{\beta}(\sigma) = \hat{\beta}_+(\sigma) + \hat{\beta}_-(\sigma)$$

from now on I will drop ^

$$T(\sigma) = \frac{1}{4}\left(\beta_+(\sigma)^2 + 2\beta_-(\sigma)\beta_+(\sigma) + \beta_-(\sigma)^2 \right)$$

Theorem.

$$[T(\sigma), T(\sigma')] = \hbar\left(T(\sigma) + T(\sigma')\right)\delta'(\sigma-\sigma') +$$

$$+ \frac{4\hbar^2 i}{3\pi}\delta^{(3)}(\sigma-\sigma')$$

As we will see Fourier comp. of $T(\sigma)$ generate centrally extended Witt algebra (Virasoro algebra)

Proof. H.W.

Claim

$$[T(\sigma), \beta(\sigma')] = \hbar\,\beta(\sigma)\,\delta'(\sigma-\sigma')$$

its Fourier components commute as

$$[L_n, \beta(\sigma')] = in\frac{\hbar}{2\pi}\beta(\sigma)e^{-in\sigma'} - \frac{\hbar}{2\pi}\partial_{\sigma'}\beta(\sigma')e^{-in\sigma'}$$

Next time:
- quantum field theory for cylinders
- OPE
- Virasoro algebra.