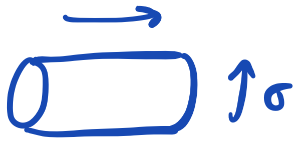
Lect 15

Free Bosons on a cylinder



$-\pi \le \sigma \le \pi$ $\qquad d^2s = d\sigma^2 + d\tau^2$

$\tau_1 \le \tau \le \tau_2$

$$\xi = \sigma + i\tau, \quad \bar{\xi} = \sigma - i\tau$$

$$\partial_\sigma = \partial_\xi + \partial_{\bar\xi}, \quad \partial_\tau = i\partial_\xi - i\partial_{\bar\xi}$$

$$\partial_\xi = \tfrac{1}{2}(\partial_\sigma - i\partial_\tau), \quad \partial_{\bar\xi} = \tfrac{1}{2}(\partial_\sigma + i\partial_\tau)$$

$$S'[\pi,\varphi] = \iint \left(\pi^i\partial_i\varphi - \tfrac{1}{2}\pi^i\pi^j g_{ij}\right)\sqrt{g}\, d^2x =$$

$$= \int_{\tau_1}^{\tau_2}\int_0^{2\pi} \underline{\left(\pi^\sigma\partial_\sigma\varphi + \pi^\tau\partial_\tau\varphi - \tfrac{1}{2}\vec{\pi}^2\right)d\sigma\, d\tau}$$

$$\delta S' = \underset{(1)}{EL} + \int_{S^1} \underset{(2)}{\pi^\tau\delta\varphi\, d\sigma}\Big|_{\tau_1}^{\tau_2}$$

(1) Euler Lagrange equations

$$\begin{cases} \partial_\sigma\varphi - \pi^\sigma = 0 \\ \partial_\tau\varphi - \pi^\tau = 0 \\ \partial_\sigma\pi^\sigma + \partial_\tau\pi^\tau = 0 \end{cases} \longrightarrow \begin{cases} \underline{\partial_\sigma\varphi = \pi^\sigma} \\ \underline{\partial_\tau\pi^\tau = -\partial^2_\sigma\varphi} \end{cases}$$

$$(\partial^2_\sigma + \partial^2_\tau)\varphi = 0$$

The general solution: $\qquad (\pi = \pi^\tau)$

$$\varphi(\sigma,\tau) = a(\sigma+i\tau) + \overline{a(\sigma+i\tau)}$$

$$\pi(\sigma,\tau) = i\left(a'(\sigma+i\tau) - \overline{a'(\sigma+i\tau)}\right)$$

$$\underline{\varphi(\sigma,\tau) = a(\xi) + \overline{a(\xi)}, \quad \pi(\sigma,\tau) = i\left(a'(\xi) - \overline{a'(\xi)}\right)}$$

(2) gives 1-form

$$\alpha_{S^1} = \int_{S^1} \pi\, \delta\varphi\, d\sigma$$

on the space of boundary fields (for each comp. of ∂C) F_{S^1}, $(\pi, \varphi): S^1 \to \mathbb{R}$

$$\omega_{S^1} = \delta\alpha_{S^1} = \int_{S^1} \delta\pi \wedge \delta\varphi\, d\sigma$$

Poisson brackets ($\pi(\sigma)$, $\varphi(\sigma)$ as coord. "fncns")

$$\{\pi(\sigma), \pi(\sigma')\} = \{\varphi(\sigma), \varphi(\sigma')\} = 0$$

$$\{\pi(\sigma), \varphi(\sigma')\} = \delta(\sigma - \sigma')$$

$$\delta(\sigma) = \sum_n e^{in\sigma}, \quad \begin{cases} f(\sigma) = \sum_n \hat{f}_n e^{in\sigma} \\ \hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\sigma) e^{-in\sigma} d\sigma \\ f(\sigma + 2\pi) = f(\sigma) \end{cases}$$

$$\hat{\delta}_n = 1$$

(3) EL equations are Hamiltonian (Euclidean) ($\overset{\text{time}}{t} = i\tau$, source)

$$\partial_\tau \varphi = -\{H, \varphi\}$$

$$\partial_\tau \pi = -\{H, \pi\}$$

$$H = \frac{1}{2} \int_0^{2\pi} (-\pi^2 + \partial_\sigma \varphi^2)\, d\sigma$$

$$\{H, \varphi(\sigma)\} = -\int_0^{2\pi} \left(-\pi(\sigma')\{\pi(\sigma'), \varphi(\sigma)\}\right) d\sigma' =$$

$$= \int_0^{2\pi} \pi(\sigma')\,\delta(\sigma'-\sigma)\,d\sigma' = \pi(\sigma),$$

$$\{H, \pi(\sigma)\} = \frac{1}{2}\int_0^{\pi} 2\,\partial_{\sigma'}\varphi(\sigma')\,\underbrace{\{\partial_{\sigma'}\varphi(\sigma'), \pi(\sigma)\}}_{= -\partial_{\sigma'}\delta(\sigma-\sigma')}\,d\sigma'$$

$$= + \int_0^{2\pi} \partial^2_{\sigma'}\varphi(\sigma')\,\delta(\sigma-\sigma')\,d\sigma' = + \partial^2_\sigma \varphi(\sigma)$$

Stress-energy tensor

$$T^{ij} = -\frac{2}{\sqrt{g}}\,\frac{\delta S}{\delta g_{ij}},$$

For us:

$$T^{ij} = \pi^i \pi^j - \delta^{ij}\left(\pi^k \partial_k \varphi - \frac{1}{2}\vec{\pi}^2\right)$$

In terms of π & φ on solutions to EL (on shell)

$$\begin{cases} T^{\sigma\sigma} = \dfrac{(\partial_\sigma \varphi)^2}{2} - \dfrac{\pi^2}{2}, \\[2ex] T^{\sigma\tau} = \pi\,\partial_\sigma \varphi \\[2ex] T^{\tau\tau} = -T^{\sigma\sigma} \end{cases}$$

Observation:

$$\underline{H = -T^{\tau\tau} = \tfrac{1}{2}(\partial_\sigma\varphi)^2 - \pi^2},$$

The density of the Hamiltonian

<u>In complex coordinates</u>: $T^{ij} = T_j^i = T_{ij}$

$$T_{\xi\xi} = \tfrac{1}{4}(T_{\sigma\sigma} - 2i\,T_{\sigma\tau} - T_{\tau\tau}) = \tfrac{1}{4}\underbrace{(\partial_\sigma\varphi - i\pi)}_{\beta(\sigma,\tau)}{}^2$$

$$T_{\xi\bar\xi} = 0$$

$$T_{\bar\xi\bar\xi} = \overline{T_{\xi\xi}}$$

Introduce: $\beta = \partial_\sigma\varphi - i\pi$, $\bar\beta = \partial_\sigma\varphi + i\pi$

<u>Remark</u>. $\partial_\sigma\varphi(\sigma)$, $\pi(\sigma)$ 2π-periodic

$\varphi(\sigma + 2\pi) = \varphi(\sigma) + p$,

<u>Claim</u> On Euler-Lagrange equations:

$$\partial_{\bar\xi}\beta = 0, \quad \partial_\xi\bar\beta = 0$$

$\Rightarrow$ $\beta = \beta(\xi)$,

<u>Proof.</u> HW.

<u>Observation:</u> $\beta(\xi) = a'(\xi)$, $\overline{\beta(\xi)}$ = c.c.

For components of the stress energy we have:

$$T_{\xi\xi} = T(\xi) = \frac{1}{4}\beta(\xi)^2 = a'(\xi)^2$$

$$T_{\xi\bar{\xi}} = 0$$

$$T_{\bar{\xi}\bar{\xi}} = \overline{T(\xi)}$$

Poisson structure cont'd

$$\varphi(\sigma+2\pi) = \varphi(\sigma) + p \,,$$

$$\varphi(\sigma) = \sigma p + \sum_n \varphi_n e^{in\sigma} \,, \quad p \in \mathbb{R} \,, \quad \overline{\varphi_n} = \varphi_{-n}$$

$$\pi(\sigma) = \sum_n \pi_n e^{in\sigma} \,, \quad \overline{\pi_n} = \pi_{-n}$$

$$\delta(\sigma) = \frac{1}{2\pi} \sum_n e^{in\sigma} \,, \quad 1 = \int_{-\pi}^{\pi} \delta(\sigma) d\sigma = \sum_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\sigma} d\sigma \underset{n=0}{=} \frac{1}{2\pi} \cdot 2\pi = 1$$

$$\{p, \varphi_n\} = 0 \,, \{p, \pi_n\} = 0$$

$$\{\varphi_n, \varphi_m\} = \{\pi_n, \pi_m\} = 0$$

$$\{\pi_n, \varphi_m\} = \frac{1}{2\pi} \delta_{n,-m} \,,$$

Proof. H.W.

$$\beta(\sigma) = \partial_\sigma \varphi - i\pi \,, \quad \overline{\beta(\sigma)} - \text{c.c.}$$

Claim :

$$\{\beta(\sigma), \beta(\sigma')\} = 2i\,\delta'(\sigma-\sigma')$$

$$\{\beta, \bar\beta\} = 0$$

$$\{\overline{\beta(\sigma)}, \overline{\beta(\sigma')}\} = -2i\,\delta'(\sigma-\sigma')$$

Proof.

$$\{\beta(\sigma), \beta(\sigma')\} = -i\{\partial_\sigma\varphi(\sigma), \pi(\sigma')\} - i\{\pi(\sigma), \partial_{\sigma'}\varphi(\sigma')\}$$

$$= i\partial_\sigma\delta(\sigma'-\sigma) - i\,\partial_{\sigma'}\delta(\sigma-\sigma') = i\underset{\overset{\|}{\delta'(\sigma-\sigma')}}{\partial_\sigma\,\delta(\sigma-\sigma')} - i\underset{\overset{\|}{-\delta'(\sigma-\sigma')}}{\partial_{\sigma'}\delta(\sigma-\sigma')}$$

$$= 2i\,\delta'(\sigma-\sigma')$$

. . .

Poisson bracket between Fourier components of β :

$$\beta_n = i(n\varphi_n - \pi_n), \quad \beta_0 = p - i\pi_0$$

$$\bar\beta_n = -i(\overline{n\varphi_n} - \bar\pi_n), \quad \bar\beta_0 = p + i\pi_0$$

$$\beta(\sigma) = \sum_n \beta_n e^{in\sigma}, \quad \overline{\beta(\sigma)} = \sum_n \bar\beta_n e^{-in\sigma}$$

Claim

$$\{\beta_n, \beta_m\} = -\frac{n}{\pi}\delta_{n,-m}$$

$$\{\beta_n, \bar\beta_m\} = 0$$

$$\{\bar\beta_n, \bar\beta_m\} = -\frac{n}{\pi}\delta_{n,-m}$$

For the stress-energy tensor:

$$T(s) = \frac{1}{4}\beta(s)^2$$

$$\{T(\sigma), T(\sigma')\} = \frac{2}{16}\beta(\sigma)\{\beta(\sigma), \beta(\sigma')^2\}$$

$$= \frac{2i}{4}\beta(\sigma)\beta(\sigma')\delta'(\sigma-\sigma') . \qquad \{\beta(\sigma),\beta(\sigma')\} = 2i\delta'(\sigma-\sigma')$$

<u>Lemma</u>. As distributions (on $S^1 \times S^1$)

$$\beta(\sigma)\beta(\sigma')\delta'(\sigma-\sigma') = \frac{1}{2}(\beta(\sigma)^2 + \beta(\sigma')^2)\delta'(\sigma-\sigma')$$

<u>Proof.</u> (H.W.) $\qquad g(\sigma,\sigma') = -g(\sigma',\sigma)$ test fncn

$$\iint g(\sigma,\sigma')\,\beta(\sigma)\beta(\sigma')\delta'(\sigma-\sigma')\,d\sigma\, d\sigma' = \ldots$$

use "$\int$ by parts"

∎

<u>Corollary</u>

$$\{T(\sigma), T(\sigma')\} = i(T(\sigma) + T(\sigma'))\delta'(\sigma-\sigma')$$

$$\{T, \overline{T}\} = 0$$

$$\{\overline{T}(\sigma), \overline{T}(\sigma')\} = -i(\overline{T}(\sigma) + \overline{T}(\sigma'))\delta'(\sigma-\sigma')$$

In Fourier components

$$T(\sigma) = \sum_n L_n e^{in\sigma}, \quad L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(\sigma) e^{-in\sigma} d\sigma$$

Thm. $\{L_n, L_m\} = -\frac{n-m}{2\pi} L_{n+m}$. $(*)$

Proof. H.W.

H.W. You can also compute $(*)$ using $\{\beta_n, \beta_m\} = -\frac{n}{\pi}\delta_{n,-m}$.

Special components of the stress-energy tensor

$$L_0 = \frac{1}{4}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\beta(\sigma)^2 d\sigma = \frac{1}{8\pi}\int_{-\pi}^{\pi}(\partial_\sigma\varphi - i\pi)^2 d\sigma$$

$\bar{L}_0$ = its complex conjugate

$$L_0 + \bar{L}_0 = \frac{1}{4\pi}\int_{-\pi}^{\pi}(\partial_\sigma\varphi^2 - \pi^2)\,d\sigma = \frac{1}{2\pi} H_E$$

the Hamiltonian

$$H_E = \frac{1}{2}\int_{-\pi}^{\pi}(\partial_\sigma\varphi^2 - \pi^2)\,d\sigma$$

$$i(L_0 - \bar{L}_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\pi\,\partial_\sigma\varphi\, d\sigma = \frac{1}{2\pi}P$$

$$\{P, \pi(\sigma)\} = \partial_\sigma \pi(\sigma), \quad \{P, \varphi(\sigma)\} = \partial_\sigma\varphi(\sigma)$$

<u>Quantization</u>

For quantizing $T^*\mathbb{R}^n$ (p_i, q^i)

$\{p_i, q^j\} = \delta_i^j$,

we have: $A_\hbar = \langle \hat{p}_i, \hat{q}^j \mid \text{relations} \rangle$

$$[\hat{p}_k, \hat{q}^j] = -i\,\delta_k^j, \quad [\hat{p}_k, \hat{p}_l] = [\hat{q}^i, \hat{q}^j] = 0$$

The goal:

quantum part. fnc.

$$Z_\tau = e^{-\tau \hat{H}} : H^{S^1} \to H^{S^1}$$

or

$$Z_\tau \in \underbrace{H^{S^1} \otimes (H^{S^1})^*}_{\text{the space of boundary states.}} \simeq \mathrm{End}(H^{S^1})$$

<u>The space time category:</u> $\mathcal{C}$

Objects: S^1 (with flat metric)

Morphisms: cylinder cobordisms

The functor:

$\mathcal{C} \to \mathrm{Vect}$

- $S^1 \mapsto H^{S^1}$
- $C_\tau \mapsto \hat{Z}_\tau \in \underbrace{H^{S^1} \otimes (H^{S^1})^*}_{\text{"}H_{\partial C_\tau}\text{"}}$