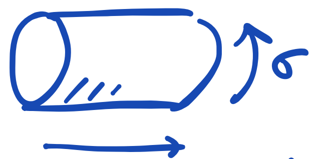
Lect 12

Last time: WZW,

$\mathcal{F}_\Sigma = C^\infty(\Sigma \to \mathbb{R})$ (noncompact, $G=\mathbb{R}$)

$$S[\varphi] = \frac{1}{2}\int_\Sigma d\varphi \wedge * d\varphi$$

Σ - cylinder,



$0 \le \sigma \le 2\pi$

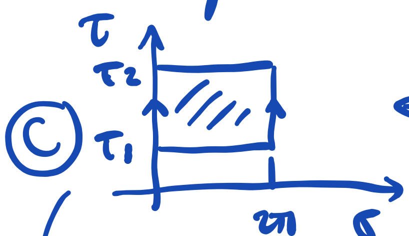
τ - Euclidean time

$\tau_1 \le \tau \le \tau_2$

$$S[\varphi] = \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^{2\pi}\left((\partial_\sigma\varphi)^2 + (\partial_\tau\varphi)^2\right) d\tau\, d\sigma$$

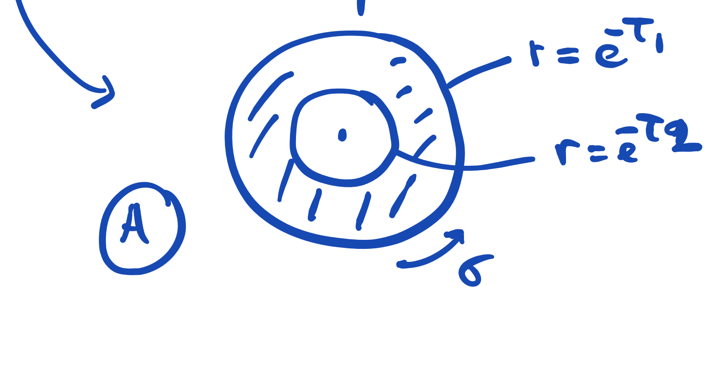
Two-dim Bose field (conformally invariant)

Complex coordinates: $\zeta = \sigma + i\tau$, $\bar{\zeta} = \sigma - i\tau$



← a cylinder

Radial complex coordinates: $z = e^{i\zeta}$



$$d\xi \wedge d\bar{\xi} = 2i \, d\sigma \wedge d\tau$$

$$\partial_\sigma \varphi = \partial_\xi \varphi + \partial_{\bar\xi} \varphi , \quad \partial_\tau \varphi = i\left(\partial_\xi \varphi - \partial_{\bar\xi}\varphi\right)$$

$$\partial_\xi = \tfrac{1}{2}\left(\partial_\sigma - i\partial_\tau\right), \quad \partial_{\bar\xi} = \tfrac{1}{2}\left(\partial_\sigma + i\partial_\tau\right)$$

From here:

$$(\partial_\sigma \varphi)^2 + (\partial_\tau \varphi)^2 = 4\, \partial_\xi \varphi \, \partial_{\bar\xi}\varphi$$

$$S'[\varphi] = -i \int_C \underline{\partial_\xi \varphi \, \partial_{\bar\xi} \varphi \; d\xi \wedge d\bar\xi} = -i \int_A \partial_z \varphi \, \partial_{\bar z}\varphi \; dz \wedge d\bar z$$

The stress energy tensor

$$T^{ij} = -\frac{2}{\sqrt{g}} \frac{\delta S'}{\delta g_{ij}} = \partial^i \varphi \, \partial^j \varphi - \underline{g^{ij} \tfrac{1}{2} \partial_k \varphi \, \partial_\ell \varphi \, g^{k\ell}}$$

For us (on C) $d^2 s = d\sigma^2 + d\tau^2$

$$\underline{T_{\sigma\sigma} = \tfrac{1}{2}(\partial_\sigma \varphi)^2 - \tfrac{1}{2}(\partial_\tau\varphi)^2, \quad T_{\tau\tau} = -T_{\sigma\sigma}}$$

$$\underline{T_{\sigma\tau} = T_{\tau\sigma} = \partial_\tau \varphi \, \partial_\sigma \varphi}$$

In holomorphic coordinates:

(not specific for Bose field, it is for any CFT)

$$\underline{T = T_{ij}\, dx^i dx^j} \in \Gamma(C, S^2(TC))$$

$$= T_{\sigma\sigma}\, d\sigma^2 + 2\, T_{\sigma\tau}\, d\sigma d\tau + T_{\tau\tau}\, d\tau^2 = \ldots \text{ HW}$$

$$= \underline{T(\xi)\,d\xi^2 + \overline{T(\xi)}\,d\bar{\xi}^2}, \quad \left(\Leftarrow \begin{array}{l} \text{symmetry} \\ \text{and } T_{11}+T_{22}=0 \end{array} \right)$$

$$T(\xi) = T_{\xi\xi} = \underline{T_{\sigma\sigma} - 2i\,T_{\sigma\tau} - T_{\tau\tau}}$$

For the free Bose theory:

$$T(\xi) = \ldots\ldots \text{HW} = (\partial_\xi \varphi)^2$$

$$\overline{T(\xi)} = (\partial_{\bar{\xi}} \varphi)^2$$

"off shell" not on solutions to EL
$T = T(\xi, \bar{\xi})$

<u>Euler-Lagrange equations:</u>

$$\boxed{\partial_\xi \partial_{\bar{\xi}} \varphi = 0},$$

the general solution:
$\varphi(\xi, \bar{\xi}) = A(\xi) + B(\bar{\xi})$

$\downarrow$

$\partial_\xi(\partial_{\bar{\xi}}\varphi) = 0 \Rightarrow$ $\boxed{\begin{array}{l} \partial_{\bar{\xi}}\varphi \text{ is antiholom} \\ \partial_{\bar{\xi}}\varphi = \text{fncn of } \bar{\xi} \end{array}}$

$\partial_{\bar{\xi}}(\partial_\xi \varphi) = 0 \Rightarrow$ $\partial_\xi \varphi$ is holom.

$T\big|_{EL} = (\partial_\xi \varphi)^2$ is a holom. fncn

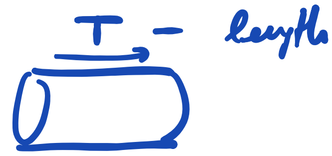
"on shell" on solutions to EL equations

$T(\xi)$ is holom, $\overline{T(\xi)}$ is antiholom.

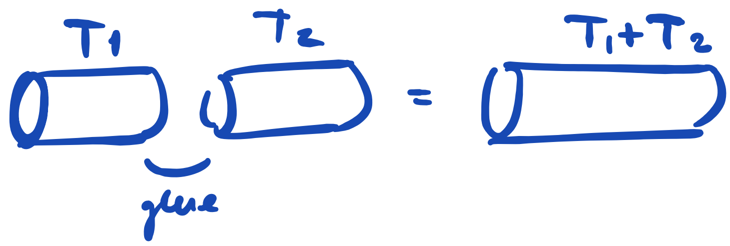
<u>Digression</u>: cyllinder space times form a Riemannian cobordism category ($d^2s = d\tau^2 + d\sigma^2$)

Objects: S^1 of radius 1

$\mathrm{Hom}(S^1, S^1) =$ cylinders,



Composition:



Hamiltonian picture

$$F^H_C = C^\infty(T^*C \to T^*\mathbb{R}), \qquad T^*\mathbb{R} \cong \mathbb{R}\times\mathbb{R}$$

$$S'[\pi,\varphi] = \iint_C \left(\pi^i \partial_i \varphi - \tfrac{1}{2}\pi^i\pi^j g_{ij}\right)\sqrt{g}\, d^2x$$

$\varphi : C \to \mathbb{R}$

$\vec{\pi}$ vector field on C

$$= \iint_C \left(\pi^\sigma \partial_\sigma \varphi + \pi^\tau \partial_\tau \varphi - \tfrac{1}{2}\left({\pi^\sigma}^2 + {\pi^\tau}^2\right)\right) d\tau\, d\sigma$$

$$\delta S' = \underset{\text{involve } \delta\vec{\pi},\ \delta\varphi}{EL} + \int_{S^1} \pi^\tau \delta\varphi\, d\sigma \Big|_{\tau_1}^{\tau_2}$$

$\pi, \delta\varphi$ are periodic in σ, $\sigma \mapsto \sigma + 2\pi$

Euler Lagrange equations

$\delta\pi^\sigma$:	$\partial_\sigma\varphi - \pi^\sigma = 0$	$\pi^\sigma = \partial_\sigma\varphi$	classical oscillator
$\delta\pi^\tau$:	$\partial_\tau\varphi - \pi^\tau = 0$	$\partial_\tau\varphi = \pi^\tau$	$\dot{q} = p$
$\delta\varphi$:	$\partial_\sigma\pi^\sigma + \partial_\tau\pi^\tau = 0$	$\partial_\tau\pi^\tau = -\partial_\sigma\pi^\sigma = -\partial_\sigma^2\varphi$	$\dot{p} = -Aq$

$\to (\partial_\sigma^2 + \partial_\tau^2)\varphi = 0$ (as in the Lagrangian picture)

Boundary terms:

on each component of $\partial C = (\{\tau_1\}\times S^1) \sqcup (\{\tau_2\}\times S^1)$

- we have boundary fields $\{\pi = \pi^\tau, \varphi : S^1 \to \mathbb{R}\} = \mathcal{F}_{S^1}$
- $\alpha_{S^1} = \int_{S^1} \pi\, \delta\varphi\, d\sigma$ (∞-dim version of $\sum_i p_i dq^i$)

  δ — de Rham diff. on $\mathcal{F}_{S^1}$
- $\Rightarrow$ $\underline{\omega_{S^1} = \delta\alpha_{S^1}} = \int_{S^1} \delta\pi \wedge \delta\varphi\, d\sigma$
- $\Rightarrow$ $(\mathcal{F}_{S^1}, \omega_{S^1})$ - $\underline{\text{exact}}$ $\overset{\infty\text{-dim}}{\text{symplectic}}$ space

$\Rightarrow$ Poisson brackets $F_1, F_2 \in$ Functionals on $\mathcal{F}_{S^1}$

$$\{F_1, F_2\} = \int_0^{2\pi} \left(\frac{\delta F_1}{\delta\pi(\sigma)} \frac{\delta F_2}{\delta\varphi(\sigma)} - (1 \leftrightarrow 2) \right) d\sigma$$

$\pi(\sigma), \varphi(\sigma)$ — consider them as local functionals ("coordinate functions" p_i, q^j)

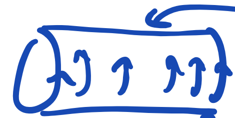
then

$$\{\pi(\sigma), \pi(\sigma')\} = 0 \quad , \{\varphi(\sigma), \varphi(\sigma')\} = 0$$

$$\{\pi(\sigma), \varphi(\sigma')\} = \delta(\sigma, \sigma')$$

EL are Hamiltonian (the Hamilt. flow) on $\mathcal{F}_{S^1}$

S^1 at $\tau = 0$



$\tau=0$ $\tau>0$

"record" of the evolution $\varphi(\sigma,\tau)$ $\pi(\sigma,\tau)$

$$\partial_\tau \varphi = \{H, \varphi\} \quad , \partial_\tau \pi = \{H, \pi\}$$

$$H = \frac{1}{2}\int_0^{2\pi} (\pi^2 - (\partial_\sigma\varphi)^2)\, d\sigma$$

(Euclidean evolution, passing to Minkowsky $\pi \mapsto \overset{?}{-i}\pi$, $\tau \mapsto it$)

$$\{H, \varphi(\sigma)\} = \frac{1}{2}\int_0^{2\pi} \{\pi^2(\sigma'), \varphi(\sigma)\} d\sigma' = \int_0^{2\pi} \pi(\sigma')\{\pi(\sigma'), \varphi(\sigma)\} d\sigma'$$

$$= \int_0^{2\pi} \pi(\sigma')\delta(\sigma,\sigma')d\sigma' = \pi(\sigma), \Rightarrow \boxed{\partial_\tau \varphi = \pi}$$

$$\{H, \pi(\sigma)\} = -\frac{1}{2}\int_0^{2\pi} \{\partial_{\sigma'}\varphi(\sigma')^2, \pi(\sigma)\} d\sigma' =$$

$$= -\frac{1}{2}\cdot 2\int_0^{2\pi} \partial_{\sigma'}\varphi(\sigma') \underbrace{\{\partial_{\sigma'}\varphi(\sigma'), \pi(\sigma)\}}_{-\partial_{\sigma'}\delta(\sigma,\sigma')} d\sigma' =$$

$$= \int_0^{2\pi} \partial_{\sigma'}\varphi(\sigma')\partial_{\sigma'}\delta(\sigma,\sigma')d\sigma' \overset{\text{def}}{=} -\int_0^{2\pi} \partial^2_{\sigma'}\varphi(\sigma')\delta(\sigma,\sigma')d\sigma'$$

$$= -\partial_\sigma^2\varphi, \quad \boxed{\partial_\tau \pi = -\partial_\sigma^2 \varphi}$$

Stress-energy tensor

$$T^{ij} \overset{\text{def}}{=} -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{ij}}$$

$$\left(\delta \det(g_{ij}) = \det(g)\cdot g^{ij}\delta g_{ij}, \Rightarrow \delta(\sqrt{g}) = \frac{\sqrt{g}}{2} g^{ij}\delta g_{ij}\right)$$

$\Rightarrow$

$$T^{ij} = \pi^i \pi^j - g^{ij}\left(\pi^k \partial_k \varphi - \frac{1}{2} g_{k\ell}\pi^k \pi^\ell\right),$$

On C

$$T^{\sigma\sigma} = \pi^{\sigma 2} - \left(\pi^\sigma \partial_\sigma \varphi + \pi^\tau \partial_\tau \varphi - \frac{1}{2}(\pi^{\sigma 2} + \pi^{\tau 2})\right) -$$

$$= \cdots \underset{\text{on } EL}{\overset{\uparrow}{\underset{}{HW}}} = \frac{(\partial_\sigma \varphi)^2}{2} - \frac{\pi^2}{2} \qquad (\pi = \pi^\tau)$$

$$T^{\sigma\tau} = \pi^\sigma \pi^\tau \underset{\text{on } EL}{\overset{}{\underset{\uparrow}{=}}} \pi\, \partial_\sigma \varphi$$

$$T^{\tau\tau} = -T^{\sigma\sigma}$$

In complex coordinates $\zeta = \sigma + i\tau$

$$\left.\begin{aligned} T_{\zeta\zeta} &= \cdots HW = \tfrac{1}{4}\underline{(\partial_\sigma \varphi + i\pi)}^2 \\ T_{\zeta\bar\zeta} &= 0 \\ T_{\bar\zeta\bar\zeta} &= \tfrac{1}{4}\underline{(\partial_\sigma \varphi - i\pi)}^2 \end{aligned}\right\} \begin{array}{l}\text{on sol's} \\ \text{to } EL\end{array}$$

<u>Remark</u>. On EL $\quad \pi = \partial_\tau \varphi$

this gives the agreement with the Lagr. result

$$T_{\zeta\zeta} = (\partial_\zeta \varphi)^2, \quad T_{\bar\zeta\bar\zeta} = (\partial_{\bar\zeta} \varphi)^2$$

Define

$$\beta(\sigma, \tau) = \partial_\sigma \varphi + i\pi^\tau$$

$$\bar\beta \qquad = \partial_\sigma \varphi - i\pi^\tau$$

On EL :

$$\partial_{\bar\zeta} \beta = 0, \quad \partial_\zeta \bar\beta = 0 \qquad (HW)$$

$\Rightarrow \quad \beta = \beta(\zeta), \quad \bar\beta = \overline{\beta(\zeta)}$ on sol's to EL

$$T_{\zeta\zeta} = \frac{1}{4}\beta(\zeta)^2, \quad T_{\bar\zeta\bar\zeta} = \frac{1}{4}\overline{\beta(\zeta)}^2$$

$\Rightarrow$ $T_{\zeta\zeta}$ is holom., $T_{\bar\zeta\bar\zeta}$ is antiholom.

$$T(\zeta) = T_{\zeta\zeta} = \frac{1}{2}\beta(\zeta)^2, \quad \overline{T(\zeta)} \; \cdots$$

<u>Poisson brackets</u>:

on S^1 (a boundary component of C)

$$\begin{cases} \varphi(\sigma) = p\sigma + \sum_n \varphi_n e^{in\sigma}, & \varphi(\sigma+2\pi) = \varphi(\sigma) + p \\ \pi(\sigma) = \sum_n \pi_n e^{in\sigma}, \end{cases}$$

φ, π are real $\Rightarrow$ $\overline{\varphi_n} = \varphi_{-n}$, $\overline{\pi_n} = \pi_{-n}$

$\{p, \varphi_n\} = 0$ $\{p, \pi_n\} = 0$, $\{\varphi_n, \varphi_m\} = 0$, $\{\pi_n, \pi_m\} = 0$

$\{\pi_n, \varphi_m\} = \delta_{n,-m}$

$$\beta(\sigma) = \partial_\sigma \varphi + i\pi = p + \sum_n in\varphi_n e^{in\sigma} + i\sum_n \pi_n e^{in\sigma}$$

on S^1

$$= \sum_n \beta_n e^{in\varphi},$$

$\beta_0 = p + i\pi$, $\beta_n = i(n\varphi_n + \pi_n)$

Similarly

$$\overline{\beta(\sigma)} = \partial_\sigma\varphi - i\pi = \sum_n \overline{\beta_{-n}}\, e^{in\sigma} = \sum_n \bar\beta_n e^{-in\sigma}$$

The goal: to compute $\{\Pi(\sigma), \Pi(\sigma')\}$

interm. goal: to compute $\{\beta, \beta\}, \{\beta, \bar\beta\} \cdots$

(i) Brackets between β, β

$$\{\beta_n, \beta_m\} = -n\{\varphi_n, \pi_m\} - m\{\pi_n, \varphi_m\} = i n \delta_{n,-m}$$

.....

<u>Next time</u>