L 11

Last time: ....

- $T(z)$, $\overline{T(z)}$ in holom. coordinates on EL equation

Examples: $g: \Sigma \to G$, G - simple, compact

- $$S_{\Sigma,g}(g) = \frac{1}{2}\int_\Sigma tr(g^{-1}dg \wedge g^{-1} * dg) =$$

invariant Ωg

$g^{ij}\sqrt{g}$ ↑ metric is conf. invariant

$$= \frac{1}{2}\int_\Sigma tr(g^{-1}\partial g \, g^{-1}\bar\partial g)\, dz \wedge d\bar z$$

$$\partial f = \frac{\partial f}{\partial z}, \quad \bar\partial f = \frac{\partial f}{\partial \bar z},$$

- $G = U(1)$, $g = e^{i\varphi}$, $\underline{\varphi \mapsto \varphi + 2\pi}$,

$$S[\varphi] = -\frac{1}{2}\int_M \partial\varphi\,\bar\partial\varphi\; dz \wedge d\bar z,$$

compactified free massless Bose field,

Wess-Zumino-Witten theory

$F_\Sigma = C^\infty(\Sigma \to G)$,

The action:

$$S_\Sigma(g) = -\frac{i}{4\pi}\overbrace{\int_\Sigma tr(g^{-1}\partial g\, g^{-1}\bar\partial g)\, dz \wedge d\bar z}^{\text{PCF part}} + WZ(g)$$

$$WZ(g) = -\frac{i}{12\pi}\int_{B_3} \mathrm{tr}\left(\Lambda^3\, \tilde{g}^{-1} d\tilde{g}\right)$$

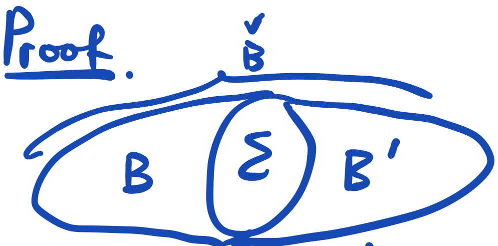
$\partial B_3 = \Sigma$, 

$\tilde{g}$ is a smooth ext. of g inside B.

B_3 = compact, oriented

<u>Proposition</u> For a fixed map $g : \Sigma \to G$, the functional $WZ(g) \bmod 2\pi i\mathbb{Z}$ does not depend on $B, \tilde{g}$.

<u>Proof</u>.



$\partial B = \Sigma \quad \partial B' = \Sigma$

$B \cup_\Sigma B'$ is a $\overbrace{\text{closed } 3\text{-manifold}}^{\text{compact, oriented}}$,

$\tilde{g}, \tilde{g}'$

$$WZ_{B,\tilde{g}}(g) - WZ_{B',\tilde{g}'}(g) = -2\pi i\left(\int_B \tilde{g}^*(\sigma) - \int_{B'} \tilde{g}'^*(\sigma)\right) = xxx$$

$$\sigma = \frac{1}{24\pi^2}\mathrm{tr}\left(G^{-1}dG \wedge G^{-1}dG \wedge G^{-1}dG\right) \in \Omega^3(G)$$

Killing form ; closed, $[\sigma] \subset H^3(G)$

$$xxx = -2\pi i \int_{\check{B}} \check{g}^*(\sigma) = \left\langle \check{g} : \check{B} \to G,\ \check{g}|_B = \tilde{g},\ \check{g}|_{B'} = \tilde{g}' \right\rangle$$

$$= -2\pi i \left\langle [\check{B}],\ \check{g}^{*}[\sigma] \right\rangle \in -2\pi i\, \mathbb{Z}$$

class of $\check{B}$ in H_3 (over $\mathbb{Z}$

the pull back to H^3 of $[\sigma] \in H^3(G)$ / over $\mathbb{Z}$

$\blacksquare$

$$S_\Sigma(g) = S^{PCF}_\Sigma(g) + WZ(g)$$

is defined mod $2\pi i\, \mathbb{Z}$

means that

$$e^{\kappa S_\Sigma(g)}$$

is the "true action". $\left(\int e^{\kappa S_\Sigma(g)} \mathcal{D}g \right.$ with some b.c. $\left. \right)$

<u>Euler-Lagrange equations</u>

$$\delta S^{PCF} = -\frac{i}{4\pi} \delta \left(\int_\Sigma tr(g^{-1}\partial g\, g^{-1}\bar\partial g)\, dz\, d\bar z \right) =$$

$(\delta(g^{-1}) = -g^{-1}\delta g\, g^{-1})$

$$= -\frac{i}{4\pi} \int_\Sigma tr\big(\underline{-g^{-1}\delta g\, g^{-1}\partial g}\, g^{-1}\bar\partial g + \underline{g^{-1}\partial \delta g\, g^{-1}\bar\partial g} +$$

$$+ \underline{g^{-1}\partial g(-g^{-1}\delta g\, g^{-1})}\bar\partial g + \underline{g^{-1}\partial g\, g^{-1}\bar\partial(\delta g)}\big)\, dz\, d\bar z =$$

$$= \frac{i}{4\pi} \int_\Sigma tr\big(g^{-1}\delta g (g^{-1}\partial g\, g^{-1}\bar\partial g) + g^{-1}\delta g (g^{-1}\bar\partial g\, g^{-1}\partial g)$$

$$- \delta g\, \partial(g^{-1}\bar\partial g\, g^{-1}) - \delta g\, \bar\partial(g^{-1}\partial g\, g^{-1})) = \text{(H.W.)}$$

$$= \frac{i}{4\pi}\int_\Sigma tr(g^{-1}\delta g(\underbrace{\partial(g^{-1}\bar\partial g)+\bar\partial(g^{-1}\partial g)}_{=0\ \sim\ \text{EL equations for PCF}}))$$

$$\delta WZ(g) = -\frac{i}{12\pi}\int_B \delta(tr(\wedge^3 \tilde g^{-1} d\tilde g)) =$$

$$= -\frac{i}{4\pi}\int_B tr(\tilde g^{-1} d\tilde g \wedge \tilde g^{-1} d\tilde g \wedge \delta(\tilde g^{-1} d\tilde g)) = \text{(H.W.)}$$

$$= -\frac{i}{4\pi}\int_B d(tr(g^{-1}\delta g\, g^{-1}dg\wedge g^{-1}dg)) =$$

$$= -\frac{i}{4\pi}\int_{\Sigma=\partial B} tr(g^{-1}\delta g\, g^{-1}dg\wedge g^{-1}dg) =$$

$$= \frac{i}{4\pi}\int_\Sigma tr(g^{-1}\delta g(\partial(g^{-1}\bar\partial g) - \bar\partial(g^{-1}\partial g)))$$

i.e.

$$\delta S^{WZW}_\Sigma(g) = \frac{i}{2\pi}\int_\Sigma tr(g^{-1}\delta g\, \partial(g^{-1}\bar\partial g)) =$$

(H.W.) $$= -\frac{i}{2\pi}\int_\Sigma tr(g^{-1}\delta g\, \bar\partial(g^{-1}\partial g)).$$

$\Rightarrow$ EL equations:

$$\partial(g^{-1}\bar\partial g)=0\,,\quad\left(\sim\ \bar\partial(g^{-1}\partial g)=0\right)$$

For $G=U(1)$, $g=e^{i\varphi}$, $\quad g^{-1}\bar\partial g = i\,\bar\partial\varphi$

EL: $\quad \partial\bar\partial\varphi=0\,,\quad \underbrace{\varphi = A(z)+\overline{A(z)}}_{\text{general solution}}$

For WZW model: $\quad g(z,\bar z):\Sigma\to G$ (compact simple…)

$$g(z,\bar z) = h_1(z)\overline{h_2(z)}\,,\quad h_1,h_2 \text{ are invertible.}$$

(SU_n; $g^* = g^{-1}$) $\quad g^* = h_2^t(\bar z)\,h_1(z)^* = \overline{h_2(z)}^{-1}h_1(z)^{-1}$

The space of solutions of EL equations has "chiral gauge symmetry"

(in quantum case the theory $\simeq L\hat{\mathfrak{g}}$ – central extention of $L\mathfrak{g}$ affine K-M algebras)

$$g(z,\bar z)\mapsto \underset{\substack{\uparrow\\ \text{holom. gauge group}}}{\Omega_1(z)}\,g(z,\bar z)\,\underset{\substack{\uparrow\\ \text{anti-holom.}}}{\overline{\Omega_2(z)}^{-1}}\,,$$