Lect 10

Last time: – Hamiltonian aspects of target symmetries

- Space-time symmetries
- stress-energy tensor (Hilbert)
- Main example scalar fields with target X

In Lagr. framework $\mathcal{F}_M = C^\infty(M \to X)$

$$S_{M,g}(\varphi) = \int_M L(\varphi, \partial\varphi)\sqrt{g}\, d^n x$$

$$e^{-\beta S}$$

In the Hamilt. framework

$$\mathcal{F}_M^H = C^\infty(T^*M \to T^*X), \quad \pi, \varphi$$

$$\varphi : M \to X, \quad \underset{\substack{\wedge \\ T^*_x M}}{\pi(t,x)} = \underbrace{\pi(x)(t)}_{\pi^a_i(x)t^i} \in T^*_{\varphi(x)} X$$

coord. on X ← coord. fields

$$S^{HJ}_{M,g}(\pi,\varphi) = \int_M \left(\pi^i_a \partial_i \varphi^a - H(\varphi,\pi)\right) \underbrace{\sqrt{g}\, d^n x}_{vol_M}$$

volume form if M is oriented

$$H(\varphi,\pi) = \max_{\partial\varphi}\left((\underset{\substack{\wedge \\ T^*_\varphi X}}{\pi}, \underset{\substack{\wedge \\ T_\varphi X}}{\partial\varphi}) - L(\underset{\substack{\wedge \\ X}}{\varphi}, \underset{\substack{\wedge \\ T_\varphi X}}{\partial\varphi})\right)$$

assume convexity so it all defined

$$\delta\left(L^{HJ}\sqrt{g}\,d^n x\right) = EL^H + d\,\underline{\alpha}^{HJ}$$

the vertical diff. in the variational bicomplex for (π, φ)

Noether form
1 - vertical form (in fields)
n-1 - horizontal form (on M)

$$\underline{\alpha}^{HJ} = \pi^i_a\,\delta\varphi^a \sqrt{g}\,\iota_{\partial_i}(d^n x),$$

it encodes boundary terms in $\delta S^{HJ}(\pi, \varphi)$

Space time symmetry is $r = r^i_{(x)}\partial_i$ a vector field on M, s.t. $N_n \subset M_n$

$$\left(\begin{array}{l} M_n = \mathbb{R}^n = \underset{\text{time}}{\mathbb{R}} \times \underset{\text{space}}{\mathbb{R}^{n-1}} \quad \Sigma \\ N_n = [t_1, t_2] \times \mathbb{R}^{n-1} \quad \Sigma \end{array} \right)$$

$$\frac{d}{dt} S_{N, R_t^*(g)}(\varphi)\Big|_{t=0} = 0, \qquad R_t = \exp(t\,r)$$

follows from HW last time

Thus. Noether currents: $\left(v_r = r^i \partial_i \varphi^a \frac{\delta}{\delta\varphi^a} + \underset{\text{H.W.}}{\overset{\pi}{\cdots}}\right)$

$$J_{v_r} = \iota_{v_r}\left(\underline{\alpha}^{HJ}\right) =$$

$$= \pi^i_a\,\underbrace{r^i \partial_i \varphi^a}\sqrt{g}\,\underbrace{\iota_{\partial_i}(d^n x)}_{\text{a form}\ (M \text{ is oriented})}$$

For $\gamma_{n-1} \subset M_n$ $\left(\begin{array}{l} M_n = [t_1, t_2] \times \Sigma_{n-1} \\ \gamma_{n-1} = \{t\} \times \Sigma_{n-1} \end{array} \right)$

$\mathcal{F}_\gamma^{HJ}$ = the restriction of fields from $\mathcal{F}_M^{HJ} = \left(\begin{smallmatrix}\text{last}\\ \text{time}\end{smallmatrix}\right)$

$= C^\infty(M \to T^*X)$, it has a natural sympl. structure (defined by sympl. str. on T^*X) (fields $= (\pi^n, \varphi)$)

$$\alpha_\gamma = \int_{\gamma_{n-1}} \underbrace{\underline{\alpha}^{HJ}}_{n-1 \text{ form}} = \int_{\gamma_{n-1}} \pi^n_a \, \delta\varphi^a \, vol_{\gamma_{n-1}}$$

$$\omega_\gamma = \delta\alpha_\gamma = \int_\gamma \delta\pi^n_a \wedge \delta\varphi^a \, vol_{\gamma_{n-1}}$$

On functional on $\mathcal{F}_\gamma^{HJ}$ we have $\{\cdot,\cdot\}_\gamma$ (last time)

Noether charge of $\gamma \subset M$ is

$$\boxed{J_{v_r}(\gamma) = \int_\gamma \pi^n_a \, r^j \partial_j \varphi^a \, vol_\gamma}$$

<u>Thm.</u> Let r be a vector field parallel to γ on γ ($r|_\gamma$ has normal component r^n, parallel means $r^n = 0$). Then v_r on $\mathcal{F}_\gamma^{HJ}$ is Hamiltonian:

$$\mathcal{L}_{v_r} F = \{J_{v_r}(\gamma), F\}, \quad F \in \text{``}C\text{''}(\mathcal{F}_\gamma^{HJ})$$

and

$$\{J_{v_r}(\gamma), J_{v_s}(\gamma)\} = J_{v_{[r,s]}}(\gamma)$$

<u>Example:</u> $M = [t_1, t_2] \times \mathbb{R}^{n-1}$,

$r = r^i \partial_i$, r - constant along $\mathbb{R}^{n-1}$

In this case $J_{v_r}(\mathbb{R}^{n-1})$ are Hamiltonians generating translation on the time slice t.

<u>Conformal invariance</u>

<u>Def</u>. A smooth map $\phi : (M, g) \to (M', g')$ is <u>conformal</u> if

$$\phi^*(g') = g\,\Omega \ , \quad \Omega > 0$$

Ω = conformal factor = Weyl factor (for ϕ)

<u>Def</u> Two metric on M that differ by a Weyl factor are called <u>conf. equivalent</u>

This equiv. class = <u>conformal structure</u> on M

<u>Claim</u>. Conformal mapping $\phi : (M, g) \circlearrowleft$

form a group $\mathrm{Conf}(M, g)$.

From now on $\dim(M) = 2$, $(M \equiv \Sigma)$

<u>Example</u>: $\exists D \subset \mathbb{R}^2$ domain, $\mathbb{R}^2 \simeq \mathbb{C}$
(x, y), $z = x + iy$
(standard compl. structure on $\mathbb{R}^2$)

Let $\phi : D \to \mathbb{R}^2$

<u>Proposition</u> (H.W.) ϕ is conformal (w.r.t. to $d^2s = d^2x + dy^2$) iff it is either holomorphic, or antiholom and has no critical points on D.
$(d\phi(z) = 0,)$

infinitesimal version

<u>Def.</u> A vector field is <u>conformal</u> if

$$\mathcal{L}_v(g) = \omega g \ , \quad \omega(x) \in \mathbb{R}$$

$$\mathrm{Conf}(M, g) \subset \mathrm{Diff}(M), \quad \mathrm{conf}(M, g) \subset \mathrm{Vect}(M)$$

the Lie alg. of conf. vector fields

all vect. fields on M

<u>Conformal vector fields on $\mathbb{R}^2$ $(d^2x + dy^2)$ $(\mathbb{C})$</u>

<u>Proposition</u> vector field $v = v_x \partial_x + v_y \partial_y$ is

conformal iff $u(x,y) = v_x(x,y) + i\, v_y(x,y)$ is holomorphic ($z = x+iy$). And

$$v = u(z)\,\partial_z + \overline{u(z)}\,\frac{\partial}{\partial \bar{z}}$$

$\Rightarrow$ Conformal vector fields on $\mathbb{R}^2$ is an ∞-dim Lie algebra.

<u>Conformal vector field on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$</u>

(z is invertible)

Basis in the space of hom. vector fields on $\mathbb{C}^*$

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z}, \quad n \in \mathbb{Z}$$

(*) $[\ell_n, \ell_m] = (n-m)\,\ell_{n+m}$ (H.W.)

<u>Proposition</u> (H.W.)

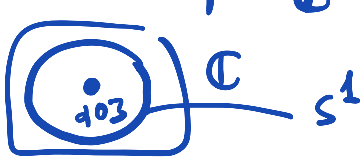
$$\mathrm{conf}(\mathbb{C}^*) = \mathrm{Span}_{\mathbb{R}}\left(\ell_n + \bar{\ell}_n,\ i(\ell_n - \bar{\ell}_n)\right)_{n\in\mathbb{Z}}$$

real Lie algebra

<u>Remark</u>. $\mathrm{conf}(\mathbb{C}^*)_{\mathbb{C}} = \mathrm{conf}(\mathbb{C}^*)\otimes_{\mathbb{R}}\mathbb{C} =$

$$= W \oplus \overline{W}$$

$W = \mathrm{Span}_{\mathbb{C}}(\ell_n)_{n\in\mathbb{Z}}$ Witt Lie algebra



Natural Lie algebra associated with S^1

$$\mathrm{Vect}(S^1) = \left\{ \sum_{n\in\mathbb{Z}} a_n e^{i\theta} \frac{\partial}{\partial\theta} \, , \; a_n = \overline{a_{-n}} \right\}$$

$\mathrm{Vect}(S^1)$ is a real form of W:

- $\sigma : \xi_n \to -\xi_{-n}$, involution

$$\sigma([\xi_n,\xi_m]) = (n-m)\,\sigma(\xi_{n+m}) = -(n-m)\,\xi_{-n-m}$$

$$[\sigma(\xi_n),\sigma(\xi_m)] = [\xi_{-n},\xi_{-m}] = -(n-m)\,\xi_{-n-m}$$

- $\sigma' : W \to W$, $\sigma'([x,y]) = [\sigma'(x),\sigma'(y)]$

$$\sigma'(\xi_n) = \sigma(\xi_n)$$

$$\sigma'(\lambda x) = \overline{\lambda}\,\sigma'(x).$$

$\mathbb{C}$-antilinear involution on W

Then $\mathrm{Vect}(S^1) \cong W^{\sigma} \subset W$

$\uparrow$

subspace of σ-fixed points in W

<u>Classical conformally invariant field theories</u>

<u>Def.</u> A classical field theory is <u>conformally invariant</u> if

$$S_{M,\Omega g}(\varphi) = S_{M,g}(\varphi)$$

(the action depends only on the conf. structure on M)

$\exists\, v \in \mathrm{conf}(M, g)$, i.e. $\mathcal{L}_v(g) = \omega g$

Thm In conformally invariant field theories

$$tr(T(x)) = \sum_{ij} g_{ij}(x) T^{ij}(x) = 0$$

Proof.

by the def. of T^{ij}

$$\frac{d}{dt} S_{N, R_t^*(g)}(\varphi)\Big|_{t=0} = -\frac{1}{2}\int_N (\mathcal{L}_v g)_{ij}\, T^{ij} vol_M$$

$$= -\frac{1}{2}\int_N \omega\, tr(T(x))\, vol_M \qquad (\dim M = 2)$$

$$\Rightarrow \quad tr(T(x)) = 0.$$

$\blacksquare$

In complex coordinates ($z = x+iy$, $\bar{z} = x - iy$)

flat metric $d^2s = dx^2 + dy^2$

$$T_{zz} = T_{11} + i T_{12} + i T_{21} - T_{22}$$

$$T_{\bar{z}\bar{z}} = \overline{T_{zz}}$$

$$T_{z\bar{z}} = T_{11} - \underbrace{i T_{12} + i T_{21}}_{} + T_{22} = tr(T) = 0$$

$$T_{\bar{z}z} = 0$$

H.W. The equation $\nabla_i T^{ij} = 0$ ($\nabla_i = \partial_i$ for flat metric) is equivalent to $\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0$

$\Rightarrow \quad T_{zz} = T(z)$ holom. fun of z

$T_{\bar z\bar z} = \overline{T(z)}$

<u>Noether currents for conformal symmetries:</u>

$$J_v(z,\bar z) = T(z)v(z) + \overline{T(z)}\,\bar v(z)$$

$$v = v(z)\frac{\partial}{\partial z} + \overline{v(z)}\frac{\partial}{\partial \bar z} \text{ - conf. vect. field.}$$

Last time:

<u>Example</u>: nonlinear σ-model in 2D

$$S_{\Sigma,g}(\varphi) = \frac{1}{2}\int_\Sigma G^{ab}(\varphi)\,\partial_i\varphi^a\,\partial_j\varphi^b g^{ij}\sqrt{g}\,d^2x$$

$g^{ij}\sqrt{g}$ is a Weyl inv. comb. in 2D

Target space (X, G)

(conf. invariance here see the last lecture)

Particular case: Principal chiral field theory

$F_M = C^\infty(M\to G)$, $X = G$, ← target space G - compact simple Lie group

$g: M\to G$ metric on G → Killing form.

$$S_{\Sigma,g}[g] = \frac{1}{2}\int_\Sigma tr(\underbrace{g^{-1}dg}_{1}\wedge \underbrace{g^{-1}*dg}_{1}) = \text{in holom local coord.}$$

$$= \frac{1}{2}\int_M tr(g^{-1}\partial g\; g^{-1}\bar\partial g)\,dz\wedge d\bar z, \quad \partial u = \frac{\partial u}{\partial z},$$

Particular case $G = U(1)$, $g = e^{i\varphi}$,

$$S_{M,g}[\varphi] = -\frac{1}{2}\int_M \partial\varphi\,\bar\partial\varphi \; dz\wedge d\bar z$$

the scalar massless free field, ($\varphi : M \to \mathbb{R}$ noncompact Boson)

(in our case $\varphi \mapsto \varphi + 2\pi$) (compactified Boson).

Next time: - WZW model

- move to the quantum case.