Conformal field theory

Material. This course is a brief introduction to conformal field theory, both classical and quantum. Here is a tentative schedule. The course will last for about three months, with approximately one section (slightly less) per week.

1 Classical and quantum field theories: the framework.

1. Local classical field theories on Riemannian space time, Lagrangian and Hamiltonian frameworks.
2. Examples: nonlinear sigma models, Principal Chiral field theory, Wess-Zumino-Witten model.
3. Symmetries in local classical field theories. Noether theorem. Noether currents. Stress-energy tensor (Hilbert stress-energy).
4. Local quantum field theory, basic framework.
5. The idea of a path integral.

2 Conformal invariance in 2D.

1. Conformal maps between Riemann surfaces.
2. Conformal maps between domains in $\mathbb{R}^2$.
3. Conformal vector fields on a Riemann surface. Conformal vector fields on domains in $\mathbb{R}^2$.
4. Conformal vector fields on $\mathbb{C}^*$, Witt algebra.
5. Riemann surfaces with boundary.
6. Complex and conformal structures on a surface with punctures.
7. Mapping class group.
8. Spin structures and spinors. Majorana and Dirac spinors.

3 Conformally invariant local classical and quantum field theories in 2D. Outline.

1. Conformally invariant local classical field theories. Energy-momentum tensor.
2. Examples: classical massless Bose field theories, non-linear massless sigma-model, classical Wess-Zumino-Witten model.
3. Conformally invariant quantum field theories. Outline.
4. Central extension of $Diff(S^1)$ (conformal anomaly) and the Virasoro algebra. Its action on boundary states.
5. The (projective) action of the mapping class group.
6. The importance of conformal field theory: statistical mechanics, models of quantum field theory, boundary theory for a topological quantum field theory.

4 Massless scalar Bose field

1. The Heisenberg algebra and corresponding representations of the Virasoro algebra.
2. Vertex operators for Bose field. Chiral vertex operators.
3. Fusion relations.
4. The partition function for a cylinder and a torus. Modular property.

5 Algebraic structure of 2D conformal field theories.

1. Elements of representation theory of the Virasoro algebra
2. Screening operators.
3. Minimal conformal field theories.

6 Vertex operator algebras.

1. Axioms of vertex operator algebras.
2. The Virasoro algebra as a vertex operator algebra. Affine Kac-Moody algebras as vertex operator algebras.
3. Lattice vertex operator algebra.

7 Elements of representation theory of affine Kac-Moody algebras

1. Loop algebras and affine Kac-Moody algebras. Elements of representation theory affine Kac-Moody algebras.
2. The Sugawara construction.
3. Wakimoto modules.

8 Fusion and modular tensor categories

1. Rational CFT
2. Braiding and fusion for primary fields.
3. Modular tensor categories.
4. Unitarity.
5. Example: MTC for group $\mathbb{Z}_n$.

9 The Wess-Zumino-Witten model.

1. Partition function and correlation functions via path integral.
2. Correlation functions on a sphere and Ward identities.
3. Primary fields and integrtwiners.
4. The Knizhnik-Zamolodchikov connection.
5. Corresponding modular tensor category.
6. The torus and the KZB connection.
7. Higher genus case.

10 The relation between 2D conformal field theories and 3D topological field theories.

1. Quantum Chern-Simons theory by path integral and its relation to the WZW theory.
2. Wilson lines and KZ equation.
3. Corresponding modular tensor category.

4. Invariants of 3-manifolds via surgery.

5. Relation to quantum groups at roots of unity.

2. Prerequisits. Basic course on functions of complex variables. Some familiarity with Lie algebras and their representations. Some basic knowledge differential geometry.

3. Literature. Here is a list of references which covers most of the material on CFT and applications.

Here are basic books on CFT:

References

[1] P. Di Francesco, P. Mathieu, D. Senechal, *Conformal eld theory.* Springer Science & Business Media, 2012.

[2] T. Kohno, *Conformal field theory and topology.* American Mathematical Soc., 2002.

[3] M. Schottenloher, *A mathematical introduction to conformal field theory.* Lect. Notes Phys, Vol. 759, Springer (2008).

More CFT books and reviews:

References

[1] A.A.Belavin, G.M.Tarnopolsky, "Introduction to string theory and conformal field theory". Phys. Atom. Nuclei 73, 848–877 (2010). https://doi.org/10.1134/S1063778810050108

[2] M. Flohr, Bits and pieces in logarithmic conformal field theory, Int. J. Mod. Phys. A18 (2003) 44974592, hep-th/0111228.

[3] J. Fuchs and C. Schweigert, Symmetries, Lie algebras and representations: A graduate course for physicists,. Cambridge, UK: University Press (1997) 438p.

[4] M. R. Gaberdiel and P. Goddard, Axiomatic conformal field theory, Commun. Math. Phys. 209 (2000) 549594, hep-th/9810019.

[5] M. R. Gaberdiel, An introduction to conformal field theory, Rept. Prog. Phys. 63 (2000) 607667, hep-th/9910156.

[6] S. V. Ketov, Conformal field theory,. Singapore, Singapore: World Scientific (1995) 486p.

[7] C. Schweigert, J. Fuchs, and J. Walcher, Conformal field theory, boundary conditions and applications to string theory, hep-th/0011109.

Here are some more mathematical books:

References

[1] P. Etingof, I. Frenkel, A. Kirillov, *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*, Mathematical Surveys and Monographs, 1998, Volume 58.

[2] G. Segal, *The definition of conformal field theory,* Differential geometrical methods in theoretical physics. Springer, Dordrecht (1988) 165–171.

On CFT and statistical mechanics:

References

[1] P. H. Ginsparg, Applied Conformal Field Theory: Lectures given at Les Houches Summer School in Theoretical Physics. Les Houches, France: Published in Les Houches Summer School (Jun 28 Aug 5, 1988) pp. 1168, arXiv:hep-th/9108028.

[2] M. Henkel, Conformal Invariance and Critical Phenomena, Springer: Texts and Monographs in Physics, 1999.

[3] C. Itzykson, H. Saleur, and J. B. Zuber, Conformal invariance and application to statistical mechanics,. Singapore, Singapore: World Scientific (1988) 979p.

On CFT and string theory:

References

[1] R. Blumenhagen, E. Plauschinn, Introduction to Conformal Field Theory, Lecture Notes in Physics, v. 779, Springer, 2009.