Lect 8

Symmetries in classical field theory

$\mathcal{F}_M$ - space of fields, $S_M : \mathcal{F}_M \to \mathbb{R}$

v - vector field on $\mathcal{F}_M$.

v is a "strong symmetry" if $\mathcal{L}_v S_M = 0$
(the action is v-invariant) (S_M constant on flow lines of v)

v is a symmetry if it preserves EL equations (up to boundary terms).

The idea: v as above, $\mathcal{L}$ - the Lagrangian form (n-form $n = \dim M$)

$$S_M[\varphi] = \int_M \mathcal{L}(\varphi(x))$$

Let $\mathcal{L}_v(\mathcal{L})$ be the Lie derivative of $\mathcal{L}$

$$\mathcal{L}_v(\mathcal{L}) = \iota_v(\underline{\underline{\delta}}\,\mathcal{L}) \; ; \qquad \underset{\text{local}}{\mathcal{L}} \in \Omega^{0,n}(\mathcal{F}\times M)$$

$\uparrow$ var. bicomplex

we know:

$$\underset{1,n}{\delta \mathcal{L}} = \underset{1,n}{EL} + d\underline{\alpha}$$

$$\underline{\alpha} \in \Omega^{1,n-1}(\mathcal{F}\times M)$$

Noether 1-form

$$\underset{0,n}{\mathcal{L}_v(\mathcal{L})} = \underset{0,n}{\iota_v(EL)} + \underset{0,n}{\iota_v(d\underline{\alpha})}$$

<u>Def</u>. v is a symmetry if

$$\underset{0,n}{\mathcal{L}_v(\mathcal{L})} = \underset{0,n}{d\Lambda_v}, \quad \Lambda_v \in \Omega^{0,n-1}(\mathcal{F}\times M)$$

The Lagrangian is v-invariant (up to a boundary term)

$$\iota_v(EL) = -\iota_v(d\underline{\alpha}) + d\Lambda_v = (\text{d} \pm)$$

$$= d(\iota_v(\underline{\alpha}) + \Lambda_v)$$

Euler Lagrange equations: $EL = 0$

$$\in \Omega^{1,n}(\mathcal{F}\times M)$$

$\to$ this means that EL equations are "preserved up to a boundary term"

without metric on M

Def. $J_v = \iota_v(\underline{\alpha}) + \Lambda_v$ is the Noether current for the symmetry v.

Remark. $J_v \in \Omega^{0,n-1}(\mathcal{F}\times M)$

$* : \Omega^{n-1} \to \Omega^1$, metric : $\Omega^1 \to$ vector field.

(after this we get $J = \sum_i J^i \partial_i$)

Chern-Simons (an example of a gauge symmetry)

fix a principal G-bundle over M_3, assume it is trivial, G - simple compact matrix group, $\mathfrak{g} = Lie(G)$ its Lie algebra, it is also matrix ($G \hookrightarrow U(V)$, $\mathfrak{g} = \mathfrak{gl}(V)$)

Fields are connections on G-bundle,

$\Omega^1(M, \mathfrak{g}) \simeq \mathcal{F}_M = \mathrm{Conn}\begin{pmatrix} G \times M \\ \downarrow \\ M \end{pmatrix}$ (as in pure Yang-Mills)

because G-bundle is trivial

$$S[A] = \int_{M_3} \left(tr\left(\frac{1}{2} \underset{1}{A} \wedge d\underset{2}{A} \right) + \frac{1}{3} tr\left(\underset{1}{A} \wedge \underset{1}{A} \wedge \underset{1}{A} \right) \right)$$

$$\mathcal{L} = \frac{1}{2} tr(A \wedge dA) + \frac{1}{3} tr(A \wedge A \wedge A),$$

$$\delta \mathcal{L} = \frac{1}{2}\operatorname{tr}(\delta A \wedge dA) - \frac{1}{2}\operatorname{tr}(A \wedge d\delta A) + $$

$$+ \operatorname{tr}(\delta A \wedge A \wedge A) = \Big| \ \operatorname{tr}(A \wedge d\delta A) =$$

$$= -d\left(\operatorname{tr}(A \wedge \delta A)\right) + \operatorname{tr}(dA \wedge \delta A)\big) =$$

$$= \frac{1}{2}\operatorname{tr}(\delta A \wedge dA) + \frac{1}{2}\operatorname{tr}(\underset{2}{dA} \wedge \underset{1}{\delta A}) \overset{\wedge}{} \operatorname{tr}(\delta A \wedge dA) - \frac{1}{2} d\left(\operatorname{tr}(A \wedge \delta A)\right) +$$

$$+ \operatorname{tr}(\delta A \wedge A \wedge A) = \operatorname{tr}(\delta A \wedge dA) - \frac{1}{2} d\left(\operatorname{tr}(A \wedge \delta A)\right)$$

$$+ \frac{1}{2}\operatorname{tr}(\delta A \wedge [A \wedge A]) = EL + d\underline{\alpha}$$

Noether current

EL form

$$\boxed{EL = \operatorname{tr}(\delta A \wedge F_A)}, \quad \boxed{\underline{\alpha} = -\frac{1}{2}\operatorname{tr}(A \wedge \delta A)} \in \Omega^{1,2}(\mathcal{F} \times M)$$

Euler Lagrange equations: $EL = 0, \ \forall \delta A$

$F_A = 0$

$$\delta S_M[A] = \underbrace{\int_{M_3} \mathrm{tr}(\delta A \wedge F_A)}_{\substack{\text{EL equations} \\ 0 \;\; \text{if } F_A = 0}} + \int_{\partial M_3} \left(-\mathrm{tr}(A \wedge \delta A)\right)$$

$$\begin{pmatrix} \text{Solutions to EL equations} \end{pmatrix} = \begin{pmatrix} \text{flat connections} \\ \text{on } G\times M \downarrow M, \; F_A = 0 \end{pmatrix}$$

infinitesimal gauge transformation:

$$\delta_{\partial_\lambda} A = d_A \lambda \, , \qquad \lambda \in \Omega^0(M, \mathfrak{g}) = C^\infty(M \to \mathfrak{g})$$

$$\mathcal{L}_{v_\lambda}(\underset{\substack{\uparrow \\ \text{functional on connections}}}{F}) = \int_M \mathrm{tr}\left(d_A \lambda \wedge \frac{\delta F}{\delta A(x)} \right) = \iota_{v_\lambda} \cdot \delta F$$

$d_A \lambda$ = the covariant differential of $\lambda = d\lambda + [A, \lambda]$

$$= (\partial_i \lambda + [A_i, \lambda])dx^i = (\nabla_i^A \lambda)dx^i \, ,$$

Global gauge transformations:

global $A \mapsto g^{-1} A g + g^{-1} dg$, $g = 1 + \varepsilon \lambda + \dots$, $\varepsilon \to 0$

$$A \mapsto A + \varepsilon \underbrace{(d\lambda + [A, \lambda])}_{d_A \lambda} + O(\varepsilon^2)$$

local (inf. small)

$$\mathcal{L}_{v_\lambda}(\mathcal{L}) = tr(d_A \lambda \wedge F_A) - tr(A \wedge d_A \lambda)$$

(substitute $d_A \lambda$ instead of δA)

HW1. $tr(d_A \lambda \wedge F_A) = d(tr(\lambda F_A))$

Hint: Bianci identity. $d_A F_A = 0$, prove it

HW2. $\mathcal{L}_{v_\lambda}(\mathcal{L}) = \frac{1}{2} d\,(tr(\lambda F_A)) \Rightarrow \Lambda_{v_\lambda} = \frac{1}{2} tr(\lambda F_A)$

As a result: inf. gauge transformations are symmetries and

$$J_{v_\lambda} = \mathrm{tr}(\lambda F_A)$$

This was an example when the symmetry is a gauge symmetry.

<u>General remark</u>. Noether currents are conserved on EL equations

$$d J_v = 0 \mod EL$$

(in Minkowsky case it is the usual conserv. of charge)

when M is Riemannian $J_v \xrightarrow{*} 1\text{-form} \xrightarrow{g} \text{vect. field } \vec{J}$

$$v \quad dJ_v = 0 \quad \longleftrightarrow \quad \mathrm{div}_{vol_M}(\bar{J}) = 0$$

$$\Big(\text{if } \omega \in \Omega^n(M), \; v \in \Gamma(M, TM), \; \mathcal{L}_v(\omega) = \mathrm{div}_\omega(v)\,\omega \Big)$$

locally 1-dim

in the Euclidean case it is the usual divergence)

Noether charge: $\gamma_{n-1} \subset M_n$ closed

$$J_v(\gamma) = \int_{\gamma_{n-1}} J_v = \int_{\gamma_{n-1}} \bar{J}^n_v \, vol_\gamma$$

Riemannian volume on γ induced by the metric on M_n.

(the flow of $\bar{J}$ through γ).

<u>Thm</u>. if γ' is a cont. deform. of γ

then $J_v(\gamma) = J_v(\gamma')$ mod EL

If we have several vector field $\{v_\alpha\}$ ~~and~~ each of them is a symmetry $\Rightarrow$ many charges & we have a Lie algebra of such symmetries

in CS, YM this is a Lie algebra of gauge transformations $\mathfrak{g}_M = C^\infty(M \to \mathfrak{g})$

with pointwise Lie bracket.

On the boundary $\exists$ a sympl. structure

$$\alpha_{\partial M} = \int_{\partial M} tr(A \wedge \delta A), \quad \omega_{\partial M} = \delta \alpha_{\partial M} = \int_{\partial M} tr(\delta A \wedge \delta A)$$

. . . . . Hamiltonian part of gauge symm.

Target space symmetry

$\mathcal{F}_M = C^\infty(M \to X)$, X – Riemannian f.d.

M – — " —

$\exists\, v$ – vect. field on X, locally, $v = \sum_a v^a(\varphi)\frac{\partial}{\partial \varphi^a}$

Local action:

scalar Lagrangian

$$S[\varphi] = \int_M \underbrace{L(\varphi, \partial\varphi)\sqrt{g}\, d^n x}_{\text{Lagrangian } n\text{-form}}$$

$$\delta \mathcal{L} = \left(\frac{\partial L}{\partial \varphi^a}\delta\varphi^a + \frac{\partial L}{\partial \varphi^a_i}\delta\varphi^a_i \right)\sqrt{g}\, d^n x ,$$

$$\delta\varphi^a_i = \partial_i \delta\varphi^a$$

$$= \left(\frac{\partial L}{\partial \varphi^a} - \nabla_i \frac{\partial L}{\partial \varphi^a_i} \right)\delta\varphi^a \sqrt{g}\, d^n x +$$

$$+\, \partial_i \left(\frac{\partial L}{\partial \varphi_i^a} \delta\varphi^a \sqrt{g}\, d^n x \right)$$

$$\mathcal{E}L = \left(\frac{\partial L}{\partial \varphi^a} - \nabla_i \frac{\partial L}{\partial \varphi_i^a} \right) \delta\varphi^a \sqrt{g}\, d^n x$$

$$\underline{\alpha} = \frac{\partial L}{\partial \varphi_i^a} \delta\varphi^a \sqrt{g}\, \imath_{\partial_i} (d^n x),$$

HW: $d\underline{\alpha} =$

Assume that v is a symmetry in a strong sense: $L(\varphi, \partial\varphi)$ is v-invariant, i.e.

$$\mathcal{L}_v \mathcal{L} = 0$$

$d\Lambda_v = 0$

(**Examples** $L(\varphi, \partial\varphi) = \frac{1}{2}(d\varphi, d\varphi)$

$X = \mathbb{R}$, $M = \mathbb{R}^2$, $\varphi \mapsto \varphi + \lambda$

L is invariant for these transform.)

Then by definition $\Lambda_v = 0$ $\quad (\mathcal{L}_v \mathcal{L} = d\Lambda_v)$

$$\Rightarrow \quad J_v = \iota_v \alpha = \frac{\partial L}{\partial \varphi^a_i} v^a(\varphi) \sqrt{g}\, \iota_{\partial_i}(d^n x)$$

is the Noether current.

<u>HW</u>. Prove that $dJ_v = 0$ mod EL.

Now: <u>target symmetries in the 1st order framework (Hamiltonian version)</u>

$$S[\pi,\varphi] = \int_M \left(\pi^i_a\, \partial_i \varphi^a - H(\pi,\varphi)\right) \overset{\sqrt{g}\, d^n x}{\overset{\shortparallel}{\mathrm{vol}_M}}$$

$$(\pi,\varphi) : T^*M \to T^*X, \quad (\pi,\varphi)(\overset{\overset{T^*_x M}{\Downarrow}}{\xi}, \overset{\in M}{x}) = (\overset{\in T^*_{\varphi(x)}X}{\pi_x(\xi)}, \overset{\in X}{\varphi(x)}) \in T^*X$$

π – locally a vector field with values in $T^{\vee}_{\varphi(x)}X$

or $\pi = \sum_{a,i} \pi^i_a(x)\, d\varphi^a\, \partial_i$ $\left(\pi^a \sim \tilde{\pi}^a \in \Omega^{n-1}(M)\right)$

$F^H_M = C^\infty(T^*M \to T^*X)$,

$\boxed{H(\pi,\varphi) = \text{the Legendre transform of } L(\varphi, \partial\varphi)}$

$\mathcal{L}(\pi,\varphi) = \pi^i_a \partial_i \varphi^a - H(\pi,\varphi)$,

Let v be a vector field on X.

Let us find how π should transform & ...

if we want v to be a ~~target~~ strong symmetry

- $\delta(\text{topological term}) = \int_M \left(\delta\pi^i_a\, \partial_i \varphi^a + \pi^i_a\, \partial_i\, \underbrace{\delta\varphi^a}_{v^a(\varphi)}\right)\sqrt{g}\, d^n x$

$$= \int_M \left(\delta\pi^i_a\, \partial_i\varphi^a + \pi^i_b\, \frac{\partial v^b(\varphi)}{\partial\varphi^a}\, \partial_i\varphi^a\right)\sqrt{g}\, d^n x$$

it is invariant of

$$\delta \pi^i_a = - \pi^i_b \frac{\partial v^b(\varphi)}{\partial \varphi^a}$$

("this a natural lift of v to T^*X")

Our local goal is to show that Noether charges generate such symm. on ∂M as Hamilt. vector fields (Hamilt. meaning of Noether charges)

<u>Next time</u>: • will finish the Hamiltonian part

- Space-time symmetries $\Rightarrow$ conformal symm.
- Classical def. of conformal invariance
- Quantum def.

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<u>Нётэровские токи</u>