Lecture 4

Classical field theory. Lagrangian (second order)
Hamiltonian (1st order)

2nd order $\to$ 1 order

$\varphi : M \to X$ $\qquad$ $(\pi, \varphi) : TM \to TX$

$\uparrow$ space time $\quad$ $\uparrow$ target $\qquad$ (π^i_a, φ^a)

Pure Yang-Mills

M – Riemannian (Euclidean version) [M – (3,1) Minkowski; " real world]

(a) $\mathcal{P}$ – a principal G-bundle over M, $\quad$ $\mathcal{P} \leftarrow G$, $\mathcal{P} \downarrow M$

G – simple, compact Lie group (SU_n)

For simplicity assume $\mathcal{P} = M \times G$ is trivial

$\mathcal{F}_M = \{ \text{connections on } \mathcal{P} \} \simeq \Omega^1(M, \mathfrak{g})$ globally

(D. Freed, Classical CS theory) $\mathfrak{g} = \mathrm{Lie}(G)$

(if P is not trivial $\mathcal{J}|_{U \subset M} \simeq \Omega^1(U, \mathfrak{g})$)

$\Omega^\bullet(M, \mathfrak{g}) = \Omega^\bullet(M) \otimes_{\mathbb{R}} \mathfrak{g}$

{ $\mathfrak{g} = T_e G$, $[\cdot,\cdot]$ induced by the group mult.

1 form $\omega(x) = \sum_i \omega_i(x) dx^i$, $\omega_i(x) \in \mathfrak{g}$

The gauge group $G_M = \mathrm{Aut}(P) = C^\infty(M \to G)$

$\uparrow$ when P is trivial

act on connections as

$\mathfrak{g}$ - matrix Lie algebra

G - matrix Lie group

$$g : A \mapsto A^g = \underbrace{g^{-1} A g}_{\mathrm{Ad}_g} + \underbrace{g^{-1} dg}_{\text{Maurer-Cartan form on } g}$$

(difference between conn. and 1-form $\omega \mapsto \omega^g = g^{-1} \omega g$)

(Conn on $M \times G$) $\simeq \Omega^1(M, \mathfrak{g})$ as a space but this is not an isom. of G_M-modules

g acts on fibers by conjugation $\begin{matrix} M \times G \\ \downarrow \\ M \end{matrix}$ $\ \overset{h(x) \in G}{\underset{x}{\subset}}$ $g(x) : h(x) \mapsto g(x) h(x) g(x)^{-1}$

(B) The curvature of A:

$$F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(M, \mathfrak{g})$$

in local coord $A = \sum_i A_i \, dx^i$,

$$dA = \sum_{i<j} (\partial_i A_j - \partial_j A_i) \, dx^i \wedge dx^j,$$

$$\frac{1}{2}[A \wedge A] = \frac{1}{2}\sum_{ij} [A_i, A_j] \, dx^i \wedge dx^j = \sum_{i<j} [A_i, A_j] \, dx^i \wedge dx^j$$

$$F_A = \sum_{i<j} (\partial_i A_j - \partial_j A_i + [A_i, A_j]) \, dx^i \wedge dx^j,$$

The Yang-Mills action:

$$S_M[A] = \frac{1}{2}\int_M \mathrm{tr}\Big(\overset{\text{2-form}}{F_A} \wedge \underset{\text{metric}}{*} \overset{n-2}{F_A} \Big) = \frac{1}{2}\int_M \mathrm{tr}((F_A, F_A)) \cdot \mathrm{vol}_M$$

otherwise Killing form G and $\mathfrak{g}$ are matrix

$$\delta S'_M[A] = \int_M \mathrm{tr}(\delta F_A \wedge * F_A) = \left\langle \overset{\text{H.W.}}{\delta F_A = d_A \delta A} \right\rangle$$

$$d_A \alpha = d\alpha + [A, \wedge \alpha]$$

$$= \int_M \mathrm{tr}(\underline{\underline{d_A}} \delta A \wedge * F_A) = - \underbrace{\int_M \mathrm{tr}(\delta A \wedge d_A * F_A)}_{EL} +$$

$$+ \underbrace{\int_{\partial M} \mathrm{tr}(\delta A \wedge * F_A)}_{\text{Noether}}$$

$$\boxed{\begin{array}{l}\text{the space of connection is an} \\ \text{affine space over 1-forms} \\ A + \underline{\alpha}, \quad A^g = g^{-1} A g + g^{-1} dg\end{array}}$$

$$\boxed{\Rightarrow \delta A \in \Omega^1(M, \mathfrak{g})} \text{ transf. as 1-form}$$

- EL: $\quad d_A * F_A = 0$
- Noether 1-form: $\quad \underline{\alpha}_A = \mathrm{tr}(\underline{\delta A} \wedge * F_A) \in \Omega^{n-1,1}(M \times F_M)$

(c) Gauge symmetry:

$$\underline{F_{A^g} = g^{-1} F_A g} \qquad \left(\begin{array}{c}\text{transforms as} \\ \text{a 2-form}\end{array}\right)$$

$$\Rightarrow \begin{cases} S_M[A^g] = S_M[A], \\ \alpha_{A^g} = \alpha_A \end{cases}$$

∞- dimensional symmetry G_M. $\left\{\right.$ EL equations are highly degenerate

(d) First order version:

Extra field (the "momentum" corresponding to "coordinate" A)

$$\pi = \sum_{i<j} \pi^{ij} \partial_i \wedge \partial_j \in \mathrm{Vect}^{(2)}(M, \mathfrak{g}) = \Gamma^\infty(\Lambda^2 TM \otimes \mathfrak{g})$$

$$H(A, \pi) = \max_{F_A} \left(\mathrm{tr}(\iota_\pi(F_A)) - \frac{1}{2}\mathrm{tr}(F_A, F_A)\right)$$

$$= \frac{1}{2}\mathrm{tr}(\pi, \pi)$$

substitution of a 2-vect. field π into 2-form F_A

in local coord:

$$\max_{(F_A)_{ij}} \left(\sum_{i<j} tr\left(\pi^{ij} (F_A)_{ij} \right) - \frac{1}{2} \sum_{\substack{ij \\ kl}} tr\left(F_{ij} F_{kl}\, g^{ik} g^{jl} \right) \right)$$

$$= \frac{1}{2}\, tr\left(\sum_{\substack{ij \\ kl}} tr\left(\pi^{ij} \pi^{kl}\, g_{ik}\, g_{jl} \right) \right)$$

$$B = \iota_{\underset{\substack{\uparrow \\ \text{2-vect fiel}}}{\pi}} \left(\underset{\substack{\uparrow \\ n}}{vol_M} \right) \quad \Big(\text{compute it in local coord.} \Big)$$

H.W.

$$B \in \Omega^{n-2}(M, \mathfrak{g})$$

$$S_M^{YM}(A, B) = \int_M \left(tr\, \iota_\pi (F_A) - \frac{1}{2} tr(\pi, \pi) \right) vol_M$$

$$= \underline{\int_M tr(B \wedge F_A)} - \frac{1}{2} \underline{\int_M tr(B \wedge * B)}$$

<u>Remark 1</u>. $G = U(1)$, how connections transform $A^g = A + d\theta$, $g = e^{i\theta}$

{Connection} $\simeq$ {1-form}

Electrodynamics $F_A = dA$,

$d_A f = df + Af = (\partial_i f + A_i f)\,dx^i$ $\quad A = \sum_i A_i dx^i$

$S[A] = \frac{1}{2}\int_M (dA, dA)\,\mathrm{vol}_M$ metric on M

EL $\to$ Euclidean version of Maxwell's equation

1st order:

$$S[A,B] = \overbrace{\int_M B \wedge dA}^{\text{top. term}} - \frac{1}{2}\int_M B \wedge * B$$

A - conn., $B \in \Omega^{n-1}(M)$, $*$ - Hodge star.

a <u>version of</u>: Hamilton-Jacobi action

$$\underbrace{\int_{[t_1,t_2]} p\,dq}_{\text{top. term}} - \int_{t_1}^{t_2} H(p(t),q(t))\,dt$$

$H = \frac{p^2}{2m}$

Remark 2

$$S'_M[A,B] = \int_M tr(B \wedge F_A)$$

1st order in B (no crit. pnts in B only)

BF theory

$$\delta_B S_M = \int_M tr(\delta B \wedge F_A)$$

$$\begin{cases} \delta_A S_M = \int_M tr(B \wedge d_A \delta A) = (-1)^{n-2} \int_M tr(d_A B \wedge \delta A) \\ + (-1)^{n-2} \int_{\partial M} tr(B \wedge \delta A) \end{cases}$$

quantization → torsions

EL: $F_A = 0$, the connection is flat

$d_A B = 0$, covariantly constant

Gauge classes of solutions = { geom. top. finite dim space }

$$\bullet \quad \delta_B S^{YM}_M[A,B] = \int_M tr(\underset{n-2}{\delta B} \wedge (\underset{2}{F_A - *B}))$$

$(p = \dot q)$

critical poin: $B^{(0)} = *F_A$ on 2-forms, $*^2 = id$? H.W.

$$S_M[A, B^{(0)}] = \frac{1}{2}\int_M tr(F_A \wedge *F_A)$$

$$\bullet \quad \delta_A S^{YM}_M[A,B] = \int_M tr(B \wedge d_A \delta A) =$$

$$= (-1)^{n-2}\int_M tr(d_A B \wedge \delta A) + (-1)^{n-2}\int_{\partial M} tr(B \wedge \delta A)$$

$$EL: \quad \begin{cases} d_A B = 0 \\ F_A = *B \quad \leftarrow \end{cases}$$

Noether 1-form: $\underline{\alpha} = tr(B \wedge \delta A)$

- look at $\mathcal{F}_{\partial M} = \{\text{connections on } P\}\big|_{\partial M} \oplus \Omega^{n-2}(\partial M, \mathfrak{g})$

on $\mathcal{F}_{\partial M}$

$\delta A, B$; δB pull-back to ∂M

$$\begin{cases} \alpha_{\partial M} = \int_{\partial M} tr(B \wedge \delta A) \in \Omega^1(\mathcal{F}_{\underline{\partial M}}) \\ \omega_{\partial M} = \delta \alpha_{\partial M} = \int_{\partial M} tr(\delta B \wedge \delta A) \end{cases}$$

$\alpha_{\partial M}$ and $\omega_{\partial M}$ are both gauge invariant

$G_{\partial M} = C^\infty(\partial M \to G)$,

H.W. The action of $G_{\partial M}$ is Hamiltonian on $(\mathcal{F}_{\partial M}, \omega_{\partial M})$

Return to the 1st order version of the scalar field theory with target space X.

$$S_M[\pi,\varphi] = \int_M \left(\pi^i_a \partial_i \varphi^a - H(\varphi,\pi)\right) vol_M$$

$$\delta_{\pi,\varphi} S_M[\pi,\varphi] = \underset{EL}{bulk} + \int_{\partial M} \pi^i_a \delta\varphi^a \, \underbrace{\iota_{\partial_i}(vol_M)}_{n-1}$$

$$\alpha_{\partial M} = \int_{\partial M} \pi^i_a \delta\varphi^a \, \iota_{\partial_i}(vol_M) = \int_{\partial M} \pi^n_a \delta\varphi^a \underbrace{\sqrt{g_\partial}\, d^{n-1}x_\partial}_{induced}$$

normal comp. to ∂M

Def ① A <u>local boundary condition</u> for EL_M is

$$L_{N_\alpha} \subset F_{N_\alpha}$$

N_α - connected components of ∂M

② Solutions to $\mathcal{EL}_M$ with b.c. $\{L_{N_\alpha}\}$ are $\varphi \in \mathcal{F}_M$

$$\mathcal{EL}_M(\varphi) = 0, \quad \varphi|_{N_\alpha \subset \partial M} \in L_{N_\alpha}$$

Def local b.c. are called variational if

$$\alpha_{\partial M}\Big|_{L_{N_\alpha}} = 0 \quad \text{(pull-back)}$$

$\Rightarrow$ if φ is a sol. to $\mathcal{EL}_M$ with var. b.c. then it is a critical point of $S_M(\varphi)$

$\exists\ N \subset \partial M$, $\quad \mathcal{F}_N = C^\infty(N \to T^*X)$, $x \mapsto (\pi^n(x) \in T^*_{\varphi(x)}X, \varphi(x))$

(on M, $\pi = \pi^i \partial_i$ on ∂M we consider only norm.comp.)

$$\alpha_N = \int_N \pi^n_a \, \delta\varphi^a \, \mathrm{vol}_N \quad , \quad \omega_N = \delta \alpha_N$$

Examples:

- (a) Dirichlet b.c. $L^\eta_N = \{ \varphi^a(x) = \eta^a(x) \, / \, \pi^n_a$ are not fixed $\}$

  obvious $\alpha_N \big|_{L^\eta_N} = 0$

- (b) $\exists$ F - function on X (mixed b.c.) ($F=0$, Neumann b.c.)

  $$L^F_N = \{ \pi^n{}_a(x) = \frac{\partial F}{\partial \varphi^a}(\varphi(x)) \}$$

  is not a variational b.c.

  $$\alpha_N \big|_{L^F_N} = \int_N \frac{\partial F}{\partial \varphi^a}(\varphi(x)) \delta \varphi^a(x) \, \mathrm{vol}_N =$$

$$= \delta \int_N F(\varphi(x))\, vol_N \neq 0$$

This is an example when the action $S_M(\varphi, \pi)$ can be modified

$$\tilde{S}_M(\varphi, \pi) = S_M(\varphi, \pi) - \int_{\partial M} F(\varphi(x))\, vol_{\partial M}$$

EL_M are the same

but now L_M^F are variational.

The solutions with b.c. L_M^F are crit. pnts of $\tilde{S}_M(\varphi, \pi)$.

Now $(\mathcal{F}_N, \omega_N)$, $L_N \subset \mathcal{F}_N$ is a variational b.c. $\alpha_N|_{L_N} = 0$, (pull-back)

$$\omega_N\big|_{L_N} = \delta\alpha_N\big|_{L_N} = \delta\left(\alpha_N\big|_{L_N}\right) = 0$$

$\Rightarrow$ $L_N \subset \mathcal{F}_N$ is an <u>isotropic subspace</u> in $(\mathcal{F}_N, \omega_N)$

<u>Def</u> Lagrangian subspace in $(\mathcal{F}_N, \omega_N)$ is a maximal isotropic subspace (for f.d. mflds)

$\boxed{\text{(for } \infty\text{-dim manifold, Lagrangian} = (\text{isotropic \& coisotropic}))}$

<u>Def.</u> <u>Lagrangian b.c.</u> $L_N \subset \mathcal{F}_N$ is a maximal variational b.c., i.e. when L_N is α_N-exact and Lagrangian.

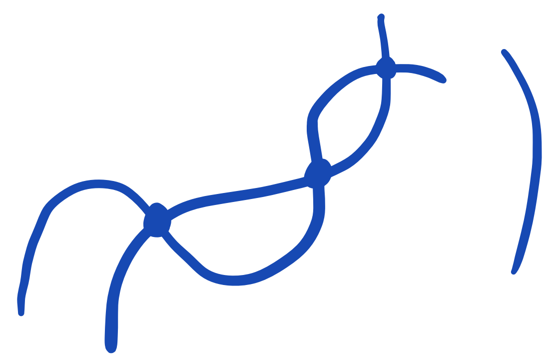
Let $EL_{\partial M} = \{$the space of boundary values of solutions to $EL_M\} \subset \mathcal{F}_{\partial M}$.

<u>Thm.</u> $EL_{\partial M}$ is an isotropic subspace

$$\omega_{\partial M}\Big|_{EL_{\partial M}} = 0$$

In good cases $EL_{\partial M}$ is a Lagrangian subspace.

For Lagrangian b.c. (2 dim.



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In a very good case

$$EL_{\partial M} \cap L_{\partial M} = \text{finitely many points}$$

= boundary values of solutions (in a very very good case we have finitely many solutions)

in a super good case $\exists!$ intersection corresponding to unique solution to EL_M with b.c. L_M. (linear PDE's with Dirichlet b.c.)

<u>Next time:</u>

- Lagrangian fibrations as b.c.
- Functorial properties of classical Hamiltonian field theory
- $\Rightarrow$ quantum field theory

. . . .

- Symmetries
- Conformal symm. in classical and quantum field theory.