Lecture 3

Examples:

① Scalar field with the target space X

$$\mathcal{F}_M = C^\infty(M \to X)$$

The space time M is a Riemannian

$\{\varphi^a\}$ local coord on X.

$$S_{M,g}(\varphi) = \int_M \underbrace{L(\varphi(x), \partial\varphi(x))}_{\text{local}} \sqrt{g(x)}\, d^n x$$

$\varphi_i^a = \partial_i \varphi^a$, $\{x^i\}$ - local coord. on M

$$\delta S_{M,g}(\varphi) = \int_M \left(\delta\varphi^a \frac{\partial L}{\partial \varphi^a} + \delta\varphi_i^a \frac{\partial L}{\partial \varphi_i^a} \right) \sqrt{g}\, d^n x$$

$$= \int_M \underline{\left(\frac{\partial L}{\partial \varphi^a} - \nabla_i \frac{\partial L}{\partial \varphi_i^a} \right)} \delta\varphi^a \sqrt{g}\, d^n x \; +$$

$$+\underbrace{\int_{\partial M} \frac{\partial L}{\partial \varphi^a_i}\, \delta \varphi^a \;\iota_{\partial_i}\left(\sqrt{g}\, d^n x\right)}$$

(M is oriented, $\sqrt{g}\, d^n x$ – the density = orientation and the volume form $\underline{\sqrt{g}\, dx^1 \wedge \dots \wedge dx^n}$)

Here ∇_i are covariant derivatives for the Levi-Civita connection (given by g_{ij}) (on TM)

$$\nabla_i f = \partial_i f, \quad \nabla_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k, \dots$$

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl}\left(\partial_i g_{lk} + \partial_k g_{lj} - \partial_l g_{ik}\right).$$

Integration by parts:

$$\int_M \partial_i f\, v^i \sqrt{g}\, d^n x = -\int_M f\, \underline{\partial_i\left(v^i \sqrt{g}\right)}\, d^n x + \underset{\text{term}}{\text{bound}}$$

H.W. 1

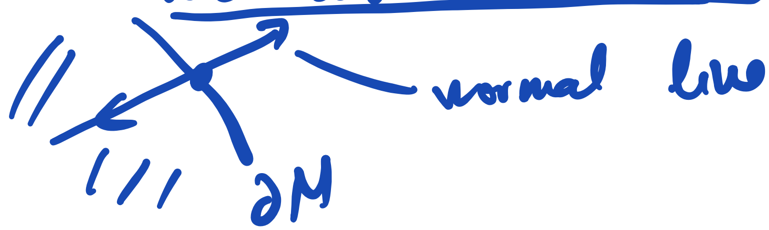
$$= -\int_M f\, \nabla_i v^i \sqrt{g}\, d^n x + \int_{\partial M} f\, v^n \underbrace{\sqrt{g_\partial}\, d^{n-1} x_\partial}$$

the normal component of v^i

the boundary Riem. volume

g_∂ = the metric induced by g on $\partial M \subset M$

2 H.W. figure out which direction is here



Euler-Lagrange equations (vanishing of the bulk term)

$$\forall (x) \quad \frac{\partial L}{\partial \varphi^a} - \nabla_i \left(\frac{\partial L}{\partial \varphi^a_i} \right) = 0$$

$$\partial_i f(\varphi(x)) = \partial_i \varphi(x) \frac{\partial f}{\partial \varphi}(\varphi(x))$$

Noether $(1, n-1)$ form $\overbrace{\text{in}}^{\text{on } \mathcal{F}_M \times M}$ the variational bicomplex

$$\underline{\alpha} = \frac{\partial L}{\partial \varphi^a_i} \underbrace{\delta\varphi^a}_{1 \text{ form on } \mathcal{F}_M} \wedge \underbrace{\iota_{\partial_i}(\sqrt{g}\, d^n x)}_{n-1 \text{ on } M}$$

The boundary term is

$$\int_{\partial M} \underline{\alpha}\Big|_{\partial M} = \int_{\partial M} \frac{\partial L}{\partial \varphi^a_n} \delta\varphi^a \sqrt{g_\partial}\, d^n x_\partial$$

the pull-back of $\underline{\alpha}$ to $\partial M \hookrightarrow M$

EL equation

$$(*) \quad \underbrace{\frac{\partial^2 L}{\partial \varphi^a_i \partial \varphi^b_j}}\, \partial_i \partial_j \varphi^b + \partial_i \varphi^b \frac{\partial^2 L}{\partial \varphi^b \partial \varphi^a_i} + \frac{\partial_i(\sqrt{g})}{\sqrt{g}} \frac{\partial L}{\partial \varphi^a_i} -$$

$$\nabla_i\left(\frac{\partial L}{\partial \varphi^a_i}\right)$$

$$-\frac{\partial L}{\partial \varphi^a} = 0,$$

H. W. 13) check this

(a) This is a second order PDE iff $\det_{a,b,i,j}\left(\frac{\partial^2 L}{\partial \varphi_i^a \partial \varphi_j^b}\right) \neq 0$

and (Euclidean classical field theory) they are elliptic if this matrix is positive definite.

<u>A particular case</u>: $M = [t_1, t_2]$, flat metric

$$S_{[t_1,t_2]}(\varphi) = \int_{t_1}^{t_2} L(\varphi, \dot\varphi)\, dt$$

The Lagrangian mechanics on X.

(b) Opposite case: $\frac{\partial^2 L}{\partial \varphi_i^a \partial \varphi_j^b} = 0$

$$\Rightarrow \quad L(\varphi, \partial\varphi) = \sum_{a,i} \alpha_a^i(\varphi)\, \partial_i \varphi^a - H(\varphi)$$

Euler-Lagr. equations:

$$\frac{\partial \alpha_a^i}{\partial \varphi^b}\partial_i \varphi^a - \underbrace{\nabla_i\left(\alpha_b^i(\varphi)\right)} - \frac{\partial H}{\partial \varphi^b} = 0,$$

$$\nabla_i\left(\alpha_b^i(\varphi)\right) = \partial_i \alpha_b^i(\varphi) + \frac{\partial_i \sqrt{g}}{\sqrt{g}}\alpha_b^i(\varphi)$$

$$\partial_i \alpha_b^i(\varphi) = \frac{\partial \alpha_b^i}{\partial \varphi^a}\partial_i \varphi^a$$

The first order terms:

$$\left(\frac{\partial \alpha_a^i}{\partial \varphi^b} - \frac{\partial \alpha_b^i}{\partial \varphi^a}\right)\partial_i \varphi^a + \dots = 0$$

1st order equation when

$$\det\left(\frac{\partial \alpha_a^i}{\partial \varphi^b} - \frac{\partial \alpha_b^i}{\partial \varphi^a}\right) \neq 0, \quad \forall i$$

Note: $\alpha_a^i(x,\varphi(x))\partial_i$ - vector field on M for each a

This gives an extra structure on M

no derivatives of φ here.

No nature α's of this sort of we will not require extra geom. str. on M.

Exception: $\dim(M)=1$, $M=[t_1,t_2]$,

$$L(\varphi,\partial\varphi)=\alpha_a(\varphi)\dot{\varphi}^a-H(\varphi),$$

Euler-Lagrange equation:

$\alpha=\alpha_a\,d\varphi^a\in\Omega^1(X)$

$$\underbrace{\left(\frac{\partial\alpha_a}{\partial\varphi^b}-\frac{\partial\alpha_b}{\partial\varphi^a}\right)}_{\omega=d\alpha}\dot{\varphi}^a=\frac{\partial H}{\partial\varphi^a},$$

$\omega=d\alpha$, nondegeneracy $\Rightarrow\det(\omega)\neq0$

solution
flow lines of the Hamilt. vector field on X generated by H.

$(X, \omega = d\alpha)$ – exact symplectic manifold.

$$S[\varphi] = \int_{t_1}^{t_2} L(\varphi, \dot{\varphi})\, dt$$ = Hamilton-Jacobi action.

$$\omega = \sum_{a,b} \left(\frac{\partial \alpha_a}{\partial \varphi^b} - \frac{\partial \alpha_b}{\partial \varphi^a} \right) d\varphi^a \wedge d\varphi^b = d\alpha$$

$$\iota_{\dot{\varphi}}(\omega) = dH, \quad \dot{\varphi} = \omega^{-1}(dH)$$

$$\omega : TM \xrightarrow{\sim} T^*M$$

<u>From 2nd order to 1st order</u>

or, naturally defined 1st order theories

$(L(\varphi, d\varphi)$, $\varphi: M \to X$, $d\varphi_{(x)}: T_xM \to T X_{\varphi(x)}$

$(\varphi, d\varphi): TM \to TX$

$$S_{M,g}(\varphi) = \int_M L(\varphi, \partial\varphi) \sqrt{g}\, d^n x ,$$

is the second order theory: $L(\varphi^{(x)}, \xi_x)$ is smooth and strictly convex function of ξ

$\xi_x \in T_{\varphi(x)} X \otimes V$, $V = T_x^* M$,

$\exists$ $H(\varphi, p)$ be the fiberwise Legendre transform of $L(\varphi, \xi)$ (in ξ).

pairing between a vect. space and its dual

$$H(\varphi, p) = \max_{\xi \in T_\varphi X \otimes V} (p(\xi) - L(\varphi, \xi))$$

exists. Because of the strict convexity

Here p is local at x

$$p \in T^*_\varphi X \otimes V^*, \qquad V^* = T_x M$$

smooth and strict convex: $\dfrac{\partial^2 L}{\partial \xi_\alpha \partial \xi_\beta}$ positive definite

Now we have the extended space of fields (φ & p).

$$\mathcal{F}^H_M = C^\infty(\underbrace{T^*M \to T^*X}_{\text{bundle maps}})$$

"global"

$$\underbrace{(\pi, \varphi)}(\underset{T^*_x M}{\underset{\cap}{\alpha}}, \overset{M}{\underset{\in}{x}}) = \Big(\underbrace{\underset{T^*_{\varphi(x)}X}{\underset{\cap}{\iota_{\pi(x)}(\alpha)}}, \ \underset{X}{\underset{\cap}{\varphi(x)}}}_{\in\, T^*X}\Big),$$

$\alpha \in T^*_x M$

$\pi(x) \in T_x M \otimes T^*_{\varphi(x)} X$

In local coordinates $\pi(x) = \pi^i(x)\partial_i$,

$\varphi(x)$ - scalar $\quad (\pi \equiv p)$

$$S^H_{M,g}(\pi,\varphi) \overset{\text{def}}{=} \int_M \Big(\underline{\iota_\pi(d\varphi)} - H(\varphi,\pi)\Big)\sqrt{g}\, d^n x$$

(natural 1st order theory because now we have fields π which are vector fields on M)

$$= \int_M \left(\pi^i_a(x)\partial_i\varphi^a - H(\varphi,\pi)\right)\sqrt{g}\, d^n x,$$

$$\delta S_{M,g}(\pi,\varphi) = \dots \qquad (\text{H.W.})$$

- Euler-Lagrange equations

$$\begin{cases} \partial_i \varphi^a - \dfrac{\partial H}{\partial \pi^i_a} = 0 \\ \nabla_i \pi^i_a + \dfrac{\partial H}{\partial \varphi^a} = 0 \end{cases}$$

- Noether 1-form:

$$\underline{\alpha} = \pi^i_a \, \delta\varphi^a \, \iota_{\partial_i}(\sqrt{g}\, d^n x)$$ $(1, n-1)$ form from the variational bicomplex

The boundary term of δS:

$$\alpha_{\partial M} = \int_{\partial M} \underline{\alpha}\Big|_{\partial M} = \int_{\partial M} \pi^n_a \, \delta\varphi^a \sqrt{g_\partial}\, d^{n-1}x_\partial$$

$\sqrt{g_\partial} = \det(g^\partial_{ij})$

1-form on $\mathcal{F}^H_{\partial M} = C^\infty(\partial M \to T^*X) =$

$x \mapsto (\pi^n(x), \varphi(x))$, $\pi^i(x) \in T^*_{\varphi(x)} X$

= "restriction of fields on M to ∂M"

$\Rightarrow$ $(\mathcal{F}^M_{\partial M}, \alpha_{\partial M})$ is an exact symplectic

∞-dim manifold, $\omega_{\partial M} = \delta \alpha_{\partial M} =$

$$= \int_{\partial M} \delta \pi_a \wedge \delta \varphi^a \, (\mathrm{vol}_\partial)$$

$\Rightarrow$ on functionals on $\mathcal{F}^M_{\partial M}$ (on local functionals)

we have the Lie algebra structure (Poisson bracket)

$$\{F, G\}_{\partial M} = \int_{\partial M} \left(\frac{\delta F}{\delta \pi^n_a} \frac{\delta G}{\delta \varphi^a} - \frac{\delta F}{\delta \varphi^a} \frac{\delta G}{\delta \pi^n_a} \right) \cdot (\mathrm{vol}_\partial)$$

Remark. The Poisson algebra of functionals on $\mathcal{F}^H_{\partial M}$ is the classical counterpart of quantum observables.

More examples

① (a) **Nonlinear σ-model with target space X**

$\begin{cases} X \text{ is a Riemannian manifold,} \\ \{G_{ab}\} - \text{the metric tensor in local coord.} \end{cases}$

$\mathcal{F}_M = C^\infty(M \to X)$ $\quad \left(ds^2_X = G_{ab}(\varphi) d\varphi^a d\varphi^b \right)$

inf. dist. on X

$$S_{M,g}(\varphi) = \int_M \frac{1}{2} g^{ij}(x) G_{ab}(\varphi) \partial_i \varphi^a \partial_j \varphi^b \cdot \sqrt{g}\, d^n x$$

$G^{ab} G_{bc} = \delta^a_c$

$$= \frac{1}{2}\int_M \langle d\varphi, d\varphi \rangle_{g,G} \cdot \sqrt{g}\, d^n x$$

Important particular case = massless free Bose field $(X = \mathbb{R})$ (massless Boson)

$ds^2 = d\varphi^2$ with const. metric

$$S_{M,g}(\varphi) = \int_M \frac{1}{2}\, g^{ij}(x)\, \partial_i \varphi\, \partial_j \varphi\, (\mathrm{vol})$$

$$= \int_M \frac{1}{2} \langle d\varphi, d\varphi \rangle_g \,\mathrm{vol}$$

(b) Same model with self interaction

$$S_{M,g}(\varphi) = \int_M \left(\frac{1}{2} \langle d\varphi, d\varphi \rangle_{g,G} - U(\varphi) \right) \mathrm{vol}_M$$

$\dim(M)=2$, $U(\varphi)=e^{\varphi}$ this is relevant to CFT (Liouville theory).

(c) The corresponding first order theory:

$$S^{H}_{M,g}(\pi,\varphi)=\int_M \left(i_{\pi}(d\varphi)-H(\varphi,\pi)\right)\mathrm{vol}_M$$

$$H(\varphi,\pi)=\max_{\xi}\left(\langle\pi,\xi\rangle-\frac{1}{2}\langle\xi,\xi\rangle\right)=\frac{1}{2}\langle\pi,\pi\rangle$$

$$=\frac{1}{2}g_{ij}(x)G^{ab}(\varphi(x))\,\pi^{i}_{a}(x)\,\pi^{j}_{b}(x)$$

$$S^{H}_{M,g}(\pi,\varphi)=\int_M\Big(\pi^{i}_{a}(x)\,\partial_i\varphi^{a}(x)-\frac{1}{2}g_{ij}(x)G^{ab}(\varphi(x))\cdot$$

$$\cdot\,\pi^{i}_{a}(x)\pi^{j}_{b}(y)\Big)\,\mathrm{vol}_M$$

($\dim M = 1$, $M = [t_1, t_2]$, this is the theory of geodesic motion in the Hamilt. framework)

For massless Bose field:

$$S^H_{M,g}(\pi, \varphi) = \int_M \left(\pi^i(x)\partial_i \varphi - \frac{1}{2} g_{ij}(x)\pi^i(x)\pi^j(x)\right) v_M$$

② <u>Principal Chiral field theory</u>

nonlinear σ-model with $X = G$,

G - simple (assume G is a matrix group

$G \subset \mathrm{Aut}(\mathbb{C}^n)$, $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathrm{End}(\mathbb{C}^n)$)

The metric on G is given by the Killing form for matrix groups $t, t' \in \mathfrak{g}$,

$$\langle t, t' \rangle = \mathrm{Tr}(t\, t')$$

For $SU(N)$, no question, it is a matrix group "on the nose". (G is compact)

$\mathcal{F}_M = C^\infty(M \to G)$

$$L(g, dg) = -\frac{1}{2}\, g^{ij}\, \mathrm{Tr}(g^{-1}\partial_i g\, g^{-1}\partial_j g)$$

$$= -\frac{1}{2}\, \mathrm{Tr}(\langle g^{-1}dg, g^{-1}dg \rangle_g)$$

<u>Note:</u> this Lagrangian is invariant with respect to $G \times G$ action $g \mapsto h\, g\, h'^{-1}$ $h, h' \in G$ (in physics $G_L \times G_R$)

H.W. Compute $H(g, \pi) \leftarrow$ the Legendre

Important particular case: $G = U(1)$, $g: M \to U(1)$

$$L(g, dg) = -\frac{1}{2} g^{ij}(x)\,(g^{-1}\partial_i g)(g^{-1}\partial_i g)$$

$\mathbb{R} \to U(1)$, $\varphi \mapsto e^{i\varphi/R}$

$U(1) \cong S^1$

R - the radius of compactification

in terms of φ:

$$L(g, dg) = \frac{1}{2} g^{ij}(x)\,\partial_i\varphi\,\partial_j\varphi \,, \quad \left(\text{same as for } X = \mathbb{R}\right)$$

but we identify $\varphi \sim \varphi + 2\pi R$,

This is the compactified Bose field.

Mass can be introduced $U(\varphi) = \frac{m^2}{2}\varphi^2$

$X = \mathbb{R}$, Bose = scalar, vector, tensor

bosons

Fermi → requires a spin structure on M,
fermions, fields = section of spin bundles on M.
(will be discussed later).

Next time: Yang-Mills, Chern-Simons,
gauge symmetry. →
conf. symm. → classical CFT.
→ quantum.

- P.Chir. field theory: classically = massless,
It is an integrable model of class. field theory ...
The quantum version is integrable &
  - massive (dynamical mass gen.)
  - asymptotically free

certain property of renormalization group which makes perturb. theory justifiable.

2D, perturbations of **CFT**.

- PCF + WZterm = WZW

  one of the most interesting CFT

  = the boundary CFT for Chern-Simons TQFT.