Lecture 2

Now: classical field theory, then
conformally inv. cl. field theory, focus on 2D

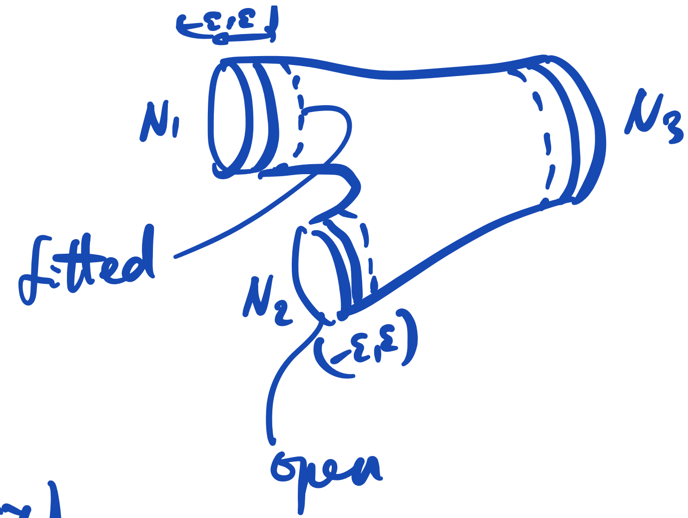
Soon: quantum field theory.

To define a classical field theory one should fix the following data:

- The category of space times (for us: n-dim (oriented) Riemannian)
  - Objects: $(n-1)$ dim smooth oriented Riemannian mflds N_{n-1} with n-dim collar with the product metric $N_{n-1} \times (-\varepsilon, \varepsilon)$, metric $= (g_N, \text{flat metric on } (-\varepsilon, \varepsilon))$
  - $\mathrm{Hom}(N, N') = \{$ n-dim oriented, smooth Riemannian manif., $\partial M = \overline{N} \sqcup N'$; with a collar that fits collars of N, N'

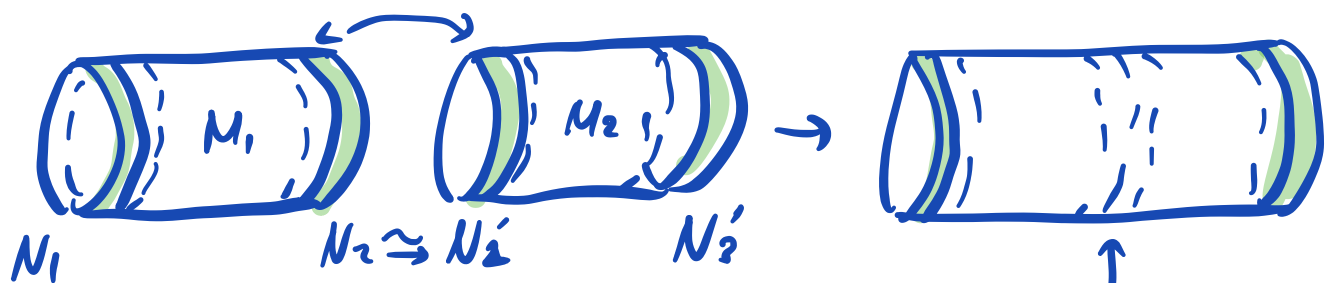
P. Teichner, S. Stolz } details

object
1-dim in the categ. of 2-dim. Riem. surfaces



(what we need for CFT)

- Composition of two morphis is the gluing. The result is smooth because the collars fit



the result of the gluing

without collars corners and nonsmoothness of the metric may occur:

topologically

Note: These are Euclidean space-times.
(Good for statistical mechanics...)

Another important category is $\underbrace{\text{Minkowski}}_{n\text{-dim}}$ cobordisms.

Objects: $(n-1)$-dim Riemannian with collars as before

Morphisms: n-dim Minkowski ($ds^2 = \overset{\text{locally}}{g_{ij}}\, dx^i dx^j$)

g has the signature $\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$ " ds^2 "

$\partial M = \overline{N} \sqcup N'$, **Example:** $M = \underset{\text{space}}{N} \times \underset{\text{time}}{[t_1, t_2]}$

Composition : Gluing,

Objects = spaces ; Morphisms = space-times

space time → ← source

• The space of fields $\mathcal{F}_M = C^\infty(M \to X)$

Example (sections of $M \times X$)
$\downarrow$
M

X - smooth oriented manifold (target space)

In local coord. on X a field is $(\varphi^1(x), \dots, \varphi^n(x))$

where $(\varphi^1, \dots, \varphi^n)$ - local coord.

• Local action functional for each M

n-form

$$S_M(\varphi) = \int_M \mathcal{L}(x; \varphi(x), \partial\varphi(x), \partial^2\varphi(x), \dots)$$

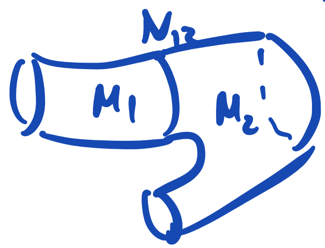
func on the jet of φ at x.

a space time
(a cobordism)

$$S_M : \mathcal{F}_M \to \mathbb{R} ,$$

defines "dynamics" (true dynamics in the Minkowski case)

$$S_{M_1 \sqcup_{N_{12}} M_2}(\varphi_{12}) = S_{M_1}(\varphi_1) + S_{M_2}(\varphi_2)$$



assuming $\varphi_1|_{N_{12}} = \varphi_2|_{N_{12}}$

$N_{12} \hookrightarrow \partial M_1$

$\hookrightarrow \partial M_2$

obvious

- The covariance: $f: M \to M'$ a diffeomorphism

(*) $$S_{M,g}(\varphi) = S_{M',(f^{-1})^* g}\left((f^{-1})^* \varphi\right)$$

This is the "nonflat, Euclidean" version of relativistic invariance in $\mathbb{R}^{3,1}$ (our space time)

Remark. The space of fields can be the space of connection, spinors or some other geom. data.

Main questions in classical field theory

- describe critical points of S_M (usually they are solution PDE) (covariance $\rightsquigarrow$ PDE are of geom. nature) | analysis

$$\Delta\varphi = 0, \ldots$$

- describe "good" boundary conditions (constraints for field on ∂M) s.t. the space of solutions = (discrete, ... "good") | analysis

v • describe how critical points (solutions to our PDE's) behave when we glue space times) | analysis,

Digression: $f : M_n \to \mathbb{R}$, $x^{(0)}$ is critical

- $df(x_0) = 0$, $\quad df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$
  in local coord.
- $\mathcal{L}_v f(x_0) = 0$, $\forall\ v \in T_{x_0} M$,

  $\quad \overset{\text{def}}{=}$

  $\frac{d}{dt} f(x(t)) \Big|_{t=0}$, $\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v \end{cases}$

the same $\quad \mathcal{L}_v f = \imath_v (df)$

locally $\quad v(x) = \sum_i v^i(x)\, \partial_i$, $\quad \imath_{\partial_i}(dx^j) = \delta_i^j$,

basis in $T_x M$ induced by $x^1, \dots, x^n$

the basis $\{dx^i\}$ in $T_x^* M$ is dual to $\{\partial_j\}$

$$\mathcal{L}_v f = \sum_i v^i(x) \frac{\partial f}{\partial x^i} = \imath_v (df)$$

<u>Similarly for local functionals</u>

$\varphi^{(0)} \in \mathcal{F}_M$ is a critical point of $S: \mathcal{F}_M \to \mathbb{R}$ if

- $\delta S = 0$, δ is the analog of de Rham differential for $\mathcal{F}_M$ (variational bicomplex)

- $\mathcal{L}_\varepsilon S(\varphi^{(0)}) = 0$,

$$\mathcal{L}_\varepsilon S(\varphi^{(0)}) = \frac{d}{dt} S(\varphi_t)\Big|_{t=0} \qquad \begin{cases} \varphi_0 = \varphi^{(0)} \\ \dot{\varphi}_0^{(x)} = \varepsilon(x) \in T_{\varphi(x)} X \end{cases} \quad x \in M$$

$$\dot{\varphi}_0^{(x)} = \frac{d}{dt} \varphi_t^{(x)}\Big|_{t=0}$$

They are equivalent.

Variational Bicomplex

$\Omega^{\bullet,\bullet}(M_1 \times M_2)$: $f(m_1, m_2)\, dx^{i_1} \wedge dx^{i_2} \cdots \otimes dy^{j_1} \wedge dy^{j_2}$

x-coord. on M_1, y-coord. on M_2

$$\Omega^{q,p}_{loc}(M \times \mathcal{F}_M) = \{q\text{-forms on } M\} \otimes \{\text{local } p\text{-forms on } \mathcal{F}_M\}$$
$$= \bigsqcup_x \{\text{expressions that depend on fields},$$

and their derivatives (jets), differentials }

$$\beta = \beta^{i_1 \cdots i_p, j_1 \cdots j_s, \cdots}(x; \varphi(x), \ldots, \varphi_{i_1 \cdots i_k}(x), \ldots)\, \delta\varphi_{i_1 \cdots i_p}(x) \wedge \delta\varphi_{j_1 \cdots j_s}(x) \wedge \cdots \in \Omega^{\bullet}(M)$$

$\otimes$

it is all at $x \in M$

- the "vertical" differential $\delta : \Omega^{p,q} \to \Omega^{p,q+1}$

$$\delta = \delta\varphi \frac{\partial}{\partial\varphi} + \delta\varphi_i \frac{\partial}{\partial\varphi_i} + \cdots + \delta\varphi_i \frac{\partial}{\partial\delta\varphi_i} + \cdots$$

$$+ \delta\varphi_{i_1 \cdots i_r} \frac{\partial}{\partial\varphi_{i_1 \cdots i_r}} + \cdots$$

Grassman variable

derivative in the Grassman algebra generated by $\delta\varphi_i, \delta\varphi_{ij}, \ldots$ (Exterior alg.)

$\circledast$

$\varphi_i = \partial_i\varphi, \quad \varphi_{ij} = \partial_i\partial_j\varphi, \ldots$

everything is at $x \in M$; $\delta\varphi_i = \frac{\partial}{\partial x^i}\delta\varphi$ at x.

- the horizontal differential (induced by the de Rham differential for M) v $\quad d = \sum_i dx^i \frac{\partial}{\partial x^i} + dx^i \varphi_i \frac{\partial}{\partial \varphi} + dx^i \varphi_{i i_1} \frac{\partial}{\partial \varphi_{i_1}} + \cdots$

$+ dx^i \varphi_{i i_1 \ldots i_r} \frac{\partial}{\partial \varphi_{i i_1 \ldots i_r}} + \cdots +$

$+ dx^i \delta\varphi_i \frac{\partial}{\partial \delta\varphi_i} + dx^i \delta\varphi_{ij_1} \frac{\delta}{\delta\varphi_{ij_1}} + \cdots$ $\left.\right\}$ all Grassm. der. are here

Grassman derivative

$\circledast$

The meaning:

forms on M

$d\left(F(x, \varphi(x), \varphi_i(x), \cdots) \, \delta\varphi(x) \wedge \delta\varphi_{i_1}(x) \wedge \cdots \right) =$

$= dx^i \frac{\partial}{\partial x^i} \left(F(x, \varphi(x), \cdots) \, \delta\varphi(x) \wedge \cdots \right) =$

= compute it using the chain rule.

Typically functionals that we consider depend on finitely many derivatives. $\Rightarrow$ finitely many terms.

An important tool

A source form does not depend on $\delta\varphi_{i_1 \cdots i_r}$

$$\omega = \omega(x,\varphi,\partial\varphi,\cdots) + \omega_a\, \delta\varphi^a + \sum_{a<b} \omega_{ab}\, \delta\varphi^a \wedge \delta\varphi^b + \cdots$$

$$\Omega^{n,1,\text{source}}_{loc} \subset \Omega^{n,1}_{loc}$$

ω & ω_a are n-forms on M

the n-forms on M, 1 order in $\delta\varphi$

Digression Grassman algebra generated by $y^1, \dots, y^m$

$$G_m = \langle y_i \mid y_i y_j = -y_j y_i \rangle,$$

$$\frac{\partial}{\partial y_i}\, y_{i_1} \cdots y_{i_k} = \delta^i_{i_1}\, y_{i_2} \cdots y_{i_k} - \delta^i_{i_2}\, y_{i_1} y_{i_3} \cdots y_{i_k} + \delta^i_{i_3}\, y_{i_1} y_{i_2} y_{i_4} \cdots$$

Grassman (left derivative)

Lemma (the decomp. lemma for source forms)

$$\Omega^{n,1}_{loc} = \Omega^{n,1,\text{source}}_{loc} \oplus d\left(\Omega^{n-1,1}_{loc}\right)$$

Proof (HW). (alg. analog of ~~integration by parts~~ Stokes thm.)

Examples:

① Classical scalar field theory with the target space X,

$\mathcal{F}_M = C^\infty(M \to X)$, M - Riemannian

$$S'_{M,g}(\varphi) = \int_M L(\varphi(x), \partial\varphi(x)) \underbrace{\sqrt{g}\, d^n x}_{\text{Riemannian volume form}}$$

in this course only such Lagrangians

$L \in C^\infty(TX)$

locally: $\sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n$

(transforms as a globally defines n-form with respect to coord. transformations)

$* : \Omega^i(M) \to \Omega^{n-i}(M)$ induced by the metric on M

(Review this) HW

$$\sqrt{g}\, d^n x = * \cdot 1$$

$$\delta S_{M,g}(\varphi) = \int_M \delta L(\varphi, \partial\varphi)\, \mathrm{vol} =$$

$$= \int_M \Big(\delta\varphi^a \frac{\partial L}{\partial \varphi^a} + \underbrace{\delta\varphi^a_i}_{= \frac{\partial \delta\varphi}{\partial x^i}} \underbrace{\frac{\partial L}{\partial \varphi^a_i}} \Big) \underbrace{\mathrm{vol}}_{= \sqrt{g}\, d^n x} =$$

⟨ also can be done in terms of the variational bicomplex (HW) (*) ⟩

$$= \int_M \Big(\delta\varphi^a \frac{\partial L}{\partial \varphi^a} - \delta\varphi^a \nabla_i \frac{\partial L}{\partial \varphi^a_i} \Big) \sqrt{g}\, d^n x +$$

(no collars here)

$$+ \int_{\partial M} \underbrace{\frac{\partial L}{\partial \varphi^a_i} \delta\varphi^a \; \iota_{\partial_i}\big(\sqrt{g}\, d^n x\big)}_{\frac{\partial L}{\partial \varphi^a_{norm}} \delta\varphi^a \sqrt{g_\partial}\, d^{n-1} y}$$

Stokes thm

$$\int_M d\Big(\frac{\partial L}{\partial \varphi^a_i} \delta\varphi^a \; \iota_{\partial_i}\big(\sqrt{g}\, d^n x\big)\Big)$$

$\frac{\partial L}{\partial \varphi^a_{norm}}$: the derivative of φ^a in the normal direction to ∂M

$\sqrt{g_\partial}$: the metric induced on ∂M

y: intrinsic coord. on ∂M

$$\delta L\, vol = \underbrace{\delta\varphi^a\left(\frac{\partial L}{\partial \varphi^a} - \nabla_i \frac{\partial L}{\partial \varphi^a_i}\right) vol}_{n} + d\Big(\underbrace{\frac{\partial L}{\partial \varphi^a_i}\delta\varphi^a\, \iota_{\partial_i}(vol)}_{n-1}\Big)$$

∇_i is the covariant derivative for Levi-Civita connection

$$\nabla_i f = \partial_i f\,, \quad (\nabla_i v)^j = \partial_i v^j + \Gamma^j_{ik} v^k\,, \ \ldots$$

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl}(\partial_i g_{lk} + \partial_k g_{lj} - \partial_l g_{ik})\,,$$

$$\delta S_{M,g}(\varphi) = 0\,,$$

EL: $$\frac{\partial L}{\partial \varphi^a} - \nabla_i \frac{\partial L}{\partial \varphi^a_i} = 0$$

Solution = critical point of S iff boundary terms are zero. This can be acheaved by imposing appropriate B.C.