Conformal field theory

1. Focus on 2D CFT :
   - applications to stat. mechanics describe universality classes of critical phenomena in 2D systems
   - building blocks of string theory
   - mathematically $\simeq$ representation theory of ∞- dim Lie groups and Lie algebras and of quantum groups
   - integrable deformations of conf. field theories =
   = relativistic (covariant) models of q.f.t. (nonperturbative) (no path integrals)

The plan:

1. Classical field theories (an outline)
   - Lagr.
   - 1-st order, or Hamilt. field theories
   - symmetries, Noether currents
   - conformal invariance
2. Outline of Q.F.T.
   - conformal invariance in QFT
3. Bose field (free, massless)
   ("Gaussian free field")
4. Representation theory of the
   Virasoro algebra.
5. . . . . . .

Classical field theory

Local, ultralocal classical field theory

To define it one should specify the following:

- "the category of space times".

Usually, the space time is the space $(\mathbb{R}^3) \times$ time $(\mathbb{R})$.

More generally: $M_n \times$ time $(\mathbb{R})$ ← space manifold

— " — : $M_{n,1}$ - pseudo Riemannian space time

[Ex. $\mathbb{R}^{3,1}$ - Minkowski space time)

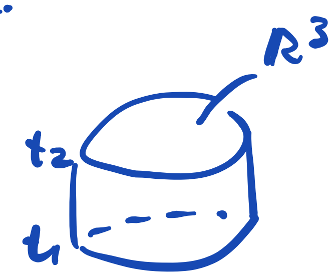
$\mathbb{R}^4$ with $\underline{\underline{d^2s = dx_1^2 + dx_2^2 + dx_3^2 - dt^2}}$

(special relativity)

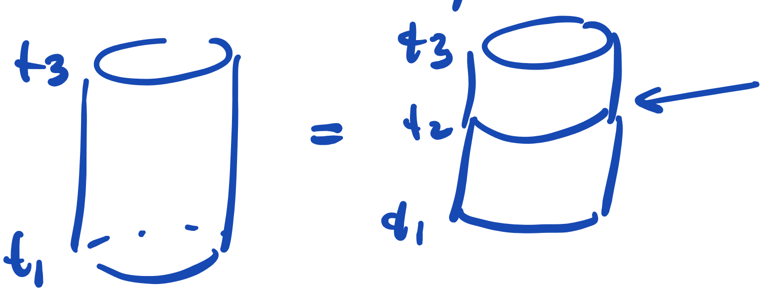
We have to be able to glue space times along a boundary:

① - <u>Minkowski</u> cylinders

$\mathbb{R}^3 \times [t_1, t_2]$



can "compose them"



← identify them along $\mathbb{R}^3 \times \{t_2\}$

the result is a category

with <u>objects</u> $= \{\mathbb{R}^3\}$

$\mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3) = \{$ all possible cylinders $\mathbb{R}^3 \times [t_1, t_2] \}$

<u>Minkowski</u> means each of the cylinders has M. metric and each object $\mathbb{R}^3$ has Euclidean metric

and

$\hookleftarrow$ 2_{t_2} of $\mathbb{R}^3$

$\hookleftarrow$ 2_{t_1} of $\mathbb{R}^3$

these embedding should be isometries.

<u>The result is an example of pseudo-Riemannian category of cobordisms</u>

② Euclidean cylinders.

the same but the metric is not Minkowski by usual Euclidean

$d^2x_1 + d^2x_2 + dx_3^2 + dx_4^2$

$(dx_4 \to i\,dt)$ (Wick's rotation)

③ The category of Riemannian 2D

manifolds (2D cobordisms)

<u>Objects</u>: $S^1 \times \dots \times S^1$, $\bigcirc \dots \bigcirc$

oriented Riemannian $\underline{\underline{dx^2}}$ in local coordinates

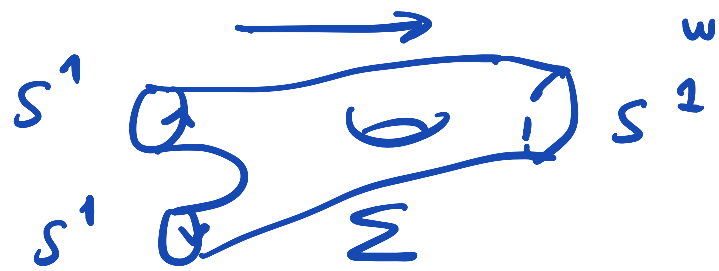
<u>Morphisms</u>: $\Sigma : (S^1)^{\times n} \to (S^1)^{\times m}$

are Riemannian 2D manifolds, oriented,

s.t. $\partial \Sigma = \partial_+ \Sigma \times \partial_- \Sigma$

$$\partial_+\Sigma \cong (S^1)^{\times n} \qquad \partial_-\Sigma \cong (S^1)^{\times m}$$

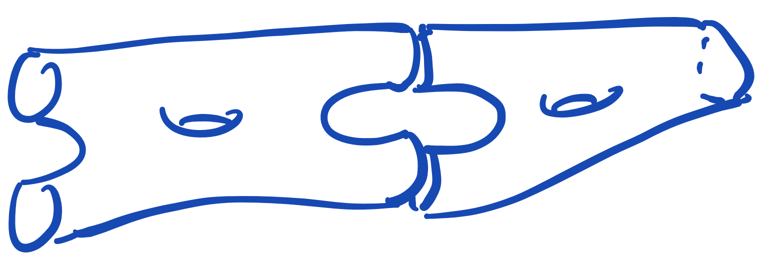
with opposite orient



$(S^1)^{\times 2} \to S^1$

<u>Compositions of morphisms</u>:

The gluing of surfaces:

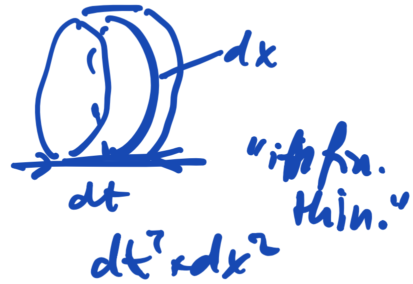


no good.

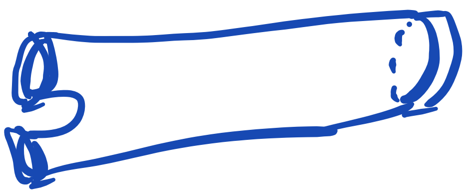
such that the result is smooth.

For this usually one assumes that

- S^1 comes with a collar
  thin open cylinder
  with the product metric



- Σ also comes with collars



<u>The result is the category of 2D Riemannian cobordisms</u>

$\boxed{\text{Our goal is to construct a quantum field theory on this category, which is conf. invariant.}}$

We start with classical field theories. To construct a classical field theory we need the following

✓ • We fix the space time

v. We need to define the space of fields.

M = space time

$$\mathcal{F}_M = \begin{cases} \text{smooth sections of } \begin{array}{c} E \leftarrow X \\ \downarrow \\ M \end{array} \ (X\text{-target space}) \\ \text{connections on a principle } G\text{-bundle over } M \\ \text{spinors} = \text{sections of a spin bundle over } M \end{cases}$$

three typical choices

- if E is trivial $E \cong M \times X$
  then $\Gamma^\infty(E) = C^\infty(M \to X)$
  X = the target space
  M = the space time

  the main example for us
- for gauge theories, gauge fields are connections (Yang Mills & Chern - Simons theories)
- relativistic fermions = spinors ...
  this important for physics

and $\exists$ important CFT with fermions, so we will discuss it

- the local action functional

$$S_M[\varphi] = \int_M \mathcal{L}(\varphi)$$

$$\mathcal{L}(\varphi) = \{L(\varphi(x), \partial\varphi(x), \dots; x)\} \in \Omega^n(M)$$

local

$n = \dim(M)$

M-oriented

$\mathcal{L}(\varphi)$ is <u>ultralocal</u> if

$$\cancel{L =} \underline{\mathcal{L}(\varphi(x), \partial\varphi(x), x)} + \cancel{d(\dots)}$$

up to ~~an~~ exact form

<u>Remark</u>. If M comes with a volume form vol then $\Omega^n(M) \ni$

$$\mathcal{L}(\varphi) = L(\varphi) \cdot vol$$

scalar functional in φ
ultralocal

In particular, if M is Riemannian, there is a Riemannian volume form

(in local coord) $\mathrm{vol} = \sqrt{g}\, d^n x$

$d^n x = dx_1 \wedge \dots \wedge dx_n$, $\quad g = \det(g_{ij})$

(coord. free) $\mathrm{vol} = * \cdot 1$

$1 \in \Omega^0(M)$, $*$ = the Hodge operation corresp. to metric on M

$$* : \Omega^i(M) \to \Omega^{n-i}(M)$$

HW: (if you do not know what $*$ is read about it)

For us M is alway Riemannian (except for the Chern-Simons, later)

$$\mathcal{L}(\varphi) = L(\varphi, \partial\varphi)\sqrt{g}\, d^n x$$

- The covariance (w.r.t. to coord. transformations) (in the flat case this is translation invariance):

$\exists\, f : M \to M'$ a diffeomorphism

$$S_{M,g}[\varphi] = S_{M', (f^{-1})^* g}[(f^{-1})^* \varphi]$$

$$M \xrightarrow{f} M'$$

metric $g \to$ metric $(f^{-1})^* g$

field $\varphi \to (f^{-1})^* \varphi$

We will be interested in such classical field theories

- critical points of $S_M[\varphi]$
- "boundary structures"

<u>Important tool</u>: variational bicomplex

(HW: de Rham differentiation on forms)

Next time: digression, differential forms in terms of Grassman raiable.

Classical field theory with fields being sections of $\begin{array}{c} E \leftarrow X \\ \downarrow \\ M \end{array}$

We assume E is trivial (locally trivial)
X - target space, smooth manifold.

$$S_{M,g}[\varphi] = \int_M L(\varphi, \overset{\text{"}d\varphi\text{"}}{\partial\varphi}) \sqrt{g}\, d^4x$$

$$\varphi \in \Gamma^\infty\left(\begin{array}{c} E \\ \downarrow \\ M \end{array}\right) \overset{\text{"}M\times X\text{"}}{=} C^\infty(M \to X)$$

Variation of S along a vector field ε on $\mathcal{F}_M$:

<u>Digression</u>. *) $f: M \to \mathbb{R}$, M - n-dim smooth manifold
v is a vector field
$v = \{v_x\}$, $v_x \in T_xM$, smooth when we vary x. Locally $v = \sum_{i=1}^{4} v^i(x) \frac{\partial}{\partial x^i}$
The directional derivative of f along v is (the Lie derivative $\mathcal{L}_v$ of f):

$$\mathcal{L}_v f = \sum_{i=1}^{n} v^i(x) \frac{\partial f}{\partial x^i}$$

**) $x_0 \in M$ is a critical point of f if for any v

$$\mathcal{L}_v f \Big|_{x=x_0} = 0$$

(directional derivative in each direction is zero)

***) $$(\mathcal{L}_v f)(x) = \frac{d}{dt} f(x(t)) \Big|_{t=0}$$

where $x(t)$ is a path with $\begin{cases} \dot{x}(0) = v(x) \\ x(0) = x \end{cases}$

The same is for functionals.

Let ε be a vector field on $\mathcal{F}_M$

φ_t - path on $\mathcal{F}_M$ s.t.

$\mathcal{F}_M$ is a vect space

$$\varphi_0 = \varphi, \quad \frac{d\varphi_t}{dt}\Big|_{t=0} = \underset{\substack{\| \\ \varepsilon \,\in\, \mathcal{F}_M}}{\varepsilon_\varphi} \in \underset{\substack{\cong \\ \mathcal{F}_M}}{T_\varphi \mathcal{F}_M}$$

Assume local coordinates $\{x^i\}$ on M, $\{\varphi^a\}$ on X

The variation of S in the direction ε (directional derivative of S)

$$\delta_\varepsilon S_{M,g}[\varphi] = \frac{d}{dt} S_{M,g}[\varphi_t]\Big|_{t=0} =$$

$$= \int_M \left(\underline{\frac{\partial L}{\partial \varphi^a}} \varepsilon^a + \frac{\partial \varepsilon^a}{\partial x^i} \underline{\underline{\frac{\partial L}{\partial \partial_i \varphi^a}}} \right) \sqrt{g}\, d^4x$$

$$= \int_M \underbrace{\left(- \frac{\partial}{\partial x^i} \left(\frac{\partial L}{\partial (\partial_i \varphi^a)} \right) + \frac{\partial L}{\partial \varphi^a} \right)}_{\to EL^a} \varepsilon^a \sqrt{g}\, d^4x$$

$$+ \int_{\partial M} \frac{\partial L}{\partial (\partial_i \varphi^a)} \langle \partial_i , \sqrt{g}\, d^4x \rangle$$

$$\langle \partial_i , \sqrt{g}\, d^4x \rangle \in \Omega^{n-1}(\partial M)$$

$$\langle \partial_i , \sqrt{g}\, d^4x \rangle = \iota_{\partial_i} (\sqrt{g}\, d^4x)$$

- Euler-Lagrange equations:

$$EL^a = 0 \quad \left(\text{a differential equation for } \varphi \right)$$

vanishes for variational b.c.

$$\langle \underline{\alpha}, \varepsilon \rangle = \int_{\partial M} \frac{\partial L}{\partial \partial_i \varphi^a} \varepsilon^a \langle \partial_i, \sqrt{g}\, d^n x \rangle$$

$\in \Omega^{n-1,1}(M \times \underline{F_M}) \ni \underline{\alpha} = \frac{\partial L}{\partial \partial_i \varphi^a} \delta \varphi^a \langle \partial_i, \cdot \rangle$

$n-1$ - form on $\overline{M}$

1 - form on F_M

Differential of φ^a

$\boxed{F_M = \text{the space of fields}}$

Noether 1-form $\in$ variational bicomplex (next time)

v - vector field on M,

locally, $v = \sum_i v^i \partial_i$

$\omega \in \Omega^k(M)$, locally $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$\langle v, \omega \rangle = \imath_v(\omega) = \sum \omega_{i_1 \dots \omega_{i_k}} (v^{i_1} dx^{i_2} \wedge \dots \wedge dx^{i_k}$$

$$+ v^{i_2} dx^{i_1} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_k} + \dots)$$

$\omega \in \Omega^1(M)$

$$\imath_v(\omega) = \sum_i \omega_i(x) v^i(x),$$

$\omega(x) \in T_x^* M$, $v(x) \in T_x M$,

$i_v(\omega)$ = evalation of the linear function ω on vector v.

Boundary conditions: $\mathcal{F}_M = C^\infty(M \to X)$ $a = 1 \dots \dim(X)$

(assuming $\frac{\partial^2 L}{\partial \partial_i \varphi \, \partial \partial_j \varphi}$ invertible) in local coord $\varphi = \{\varphi^a(x_1, .., x_n)\}$

Fix $B_{\partial M} \subset T^*\mathcal{F}_{\partial M}$ = $\left\{\begin{matrix}\text{pull back of } \mathcal{F}_M \\ \text{to } \partial M\end{matrix}\right\}$

$C^\infty(\partial M \to X)$ = $\left\{\begin{matrix}\text{restriction of fields} \\ \text{to } \partial M\end{matrix}\right\}$

Require that a solution to EL equations restricted to ∂M is in $B_{\partial M}$

Def. $B_{\partial M} \subset T^*\mathcal{F}_{\partial M}$ is a *variational* b.c. if the pull-back of α to $B_{\partial M}$ is **zero**

i.e. the boundary term is zero for all ε parallel to $B_{\partial M} \subset \mathcal{F}_{\partial M}$

ε - vector field on $\mathcal{F}_M$, $\varepsilon|_{\partial M}$ - vector field on $\mathcal{F}_{\partial M}$

(next time) (N.R. lecture on gauge systems, ~2010)

Example. $X = \mathbb{R}$, $\varphi : M \to \mathbb{R}$,

$$L(\varphi, \partial\varphi) = \frac{1}{2}\langle d\varphi, d\varphi \rangle = \frac{1}{2} \underset{\substack{\uparrow \\ \text{metric}}}{g^{ij}} \partial_i \varphi \partial_j \varphi$$

EL equations: $\Delta \varphi = 0$,

$$\Delta = \partial_i (g^{ij} \partial_j)$$

$\underline{\alpha}$ = compute (H.W.)

Noether 1-form $\;X=\mathbb{R}\;$ $\Big(T^*\mathcal{F}_{\partial M} \underset{\text{ignore topology}}{\simeq} \Omega^{n-1}(\partial M) \oplus \mathcal{F}_{\partial M}\Big)$

Dirichlet boundary conditions (fix $\eta \in C^\infty(\partial M)$)

$$B^{(\eta)}_{\partial M} = \{ \varphi(x) = \eta(x) \quad x \in \partial M \} \subset T^*\mathcal{F}_{\partial M}$$

H.W. this b.c. is variational (no restrictions on Ω^{n-1} forms)

Critical points of S are solutions to the Dirichlet problem

$$\Delta\varphi = 0 \quad x \in M, \qquad \varphi = \eta, \; x \in \partial M.$$

$X = \{$ local coordinates $\varphi^1, \dots, \varphi^k \}$ $a = 1, \dots, k$

$C^\infty(M \to X)$, $\varphi(x) \in X$

if $\varphi(x) \in U \subset X$

local coordinate neighborhood

$\varphi(x) = \{\varphi^1(x), \ldots, \varphi^K(x)\}$, $\varphi^a(x)$, $a = 1, \ldots, K$