Kastelyn matrix

$$A^K_{ij} = \begin{cases} w(ij) \,, & i \bullet\!\!\longrightarrow\!\!\bullet\, j \\ -w(ij) \,, & i \bullet\!\!\longleftarrow\!\!\bullet\, j \\ 0 \,, & \text{not connected} \end{cases}$$

<u>Theorem</u> (Kasteleyn)

$$\mathrm{Pf}(A^K) = \varepsilon^K \sum_{D \subset \Gamma} \prod_{e \in D} w(e)$$

where

$$\mathrm{Pf}(A) = \frac{1}{2^{n/2}\,(n/2)!} \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n-1} A_{\sigma_i \sigma_{i+1}}$$

A : $n \times n$, skew symmetric, n is even

and
$$\varepsilon^K = (-1)^{\sigma}\, \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{n-1}\sigma_n}$$
does not depend on the permutation σ.

<u>Proof</u>:

Dimers are perfect matchings on edges connected by edges

$$Z = \sum w(i_1 i_2) \dots w(i_{n-1} i_n)$$

<over perfect matchings $(i_1 i_2) \cdots (i_{n-1} i_n)$>

Remark. n-even $\underbrace{S_2 \times \cdots \times S_2}_{n/2} \subset S_n$ the subgroup permuting content of pairs

$$\dots (i_{k-1} i_k) \dots \leftrightarrow \dots (i_k i_{k-1}) \dots$$

$S_{n/2} \subset S_n$ the subgroup permuting pairs

$$\dots (i_{\ell-1} i_\ell) \dots (i_{k-1}, i_k) \dots \leftrightarrow \dots (i_{k-1} i_k) \dots (i_{\ell-1} i_\ell) \dots$$

As a consequence we have a bijection:

$$(\text{perfect matchings on } (1,2,\dots,n)) \leftrightarrow \left(S_n / (S_2^{n/2} \times S_{\frac{n}{2}}) \right)$$

and therefore

$$Pf(A) = \sum_{[\sigma] \in S_n / S_2^{\frac{n}{2}} \times S_{\frac{n}{2}}} (-1)^\sigma A_{\sigma_1 \sigma_2} \cdots A_{\sigma_{n-1} \sigma_n}$$

From this we can see that

$$Pf(A^K) = \sum_{D=[\sigma]} (-1)^\sigma \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{n-1}\sigma_n} \prod_{i=1}^{n-1} w(\sigma_i, \sigma_{i+1})$$

Here the sum is taken over perfect matchings on vertices on Γ connected by edges (otherwise the product of weights is zero), i.e. over dimer configurations on Γ.

Denote

$$\varepsilon^K(D) = (-1)^\sigma \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{n-1}\sigma_n}$$

where $D = [\sigma] \in S_n / S_2^{\frac{n}{2}} \times S_{\frac{n}{2}}$.

<u>Proposition</u> $\varepsilon^K(D) = (-1)^\sigma \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{n-1}\sigma_n}$ does not depend on $D = [\sigma]$. Here we assume that (σ_i, σ_{i+1}) is an edge of Γ.

__Proof__ Consider two dimer configurations D, D',

$$\varepsilon^K(D)\varepsilon^K(D') = (-1)^{\sigma}(-1)^{\sigma'}\varepsilon^K_{\sigma_1\sigma_2}\cdots\varepsilon^K_{\sigma_{n-1}\sigma_n}\cdot$$

$$\cdot\,\varepsilon^K_{\sigma'_1\sigma'_2}\cdots\varepsilon^K_{\sigma'_{n-1}\sigma'_n}. \qquad (*)$$

Let $C_1,\dots,C_m$ be connected components of composition cycles $D\cup D' \setminus$ doubly occuppied edges $= D \triangle D'$ (symmetric difference).

Choose an orientation of each C_i.

(i) The expression $(*)$ depends only on $[\sigma]=D$ and $[\sigma']=D'$ but not on representatives σ,σ'.

Let τ be the permutation shifting vertices by one edge in the direction of the orientation along each composition cycle C_i of length ≥ 4

and acting trivially on composition cycles of length two.

Clearly τ brings D' to D.

Choose σ' such that $[\sigma'] = D'$.

Define $\sigma = \tau\sigma'$.

By definition of τ, $[\sigma] = D$.

Composition cycles considered as cyclic permutations in the direction of the orientation of components define the decomposition of τ into cyclic permutations.

Therefore

$$(-1)^{\sigma}(-1)^{\sigma'} = (-1)^{\tau} = \prod_i (-1)^{\ell(C_i)+1}$$

where $\ell(C_i)$ = the length of the composition cycle C_i. Because $\ell(C_i)$ is even for all i,

$$(-1)^{\sigma}(-1)^{\sigma'}=(-1)^m$$

where m is the number of connected components in $D \Delta D'$.

For an oriented curve $C \subset \Gamma$ define $n^K(C)$ as the number of edges oriented opposite to C.

Clearly $n^k(C)+n^k(\bar{C})=|C|$. For composition cycles $|C|$ is even and therefor $n^k(C)=n^k(\bar{C})$ for composition cycles.

We have:

$$\varepsilon^K_{\sigma_1\sigma_2}\cdots\varepsilon^K_{\sigma_{n-1}\sigma_n}\varepsilon^K_{\sigma_1'\sigma_2'}\cdots\varepsilon^K_{\sigma_{n-1}'\sigma_n'}=$$

$$=\prod_{j=1}^{m}(-1)^{n(C_j)}$$

Therefore, we arrived to the formula:

$$\varepsilon^K(D)\varepsilon^K(D') = \prod_{i=1}^{m}(-1)^{n^K(C_i)+1}$$

So far we did not choose the orientation of composition cycles. Choose them now to be counter clockwise.

(ii) <u>Lemma</u> Let C be a closed counter clockwise oriented contour on $\Gamma \subset \mathbb{R}^2$

$$(-1)^{n^K(C)+1} = (-1)^{\#(C)}$$

where $\#(C)$ is the number of vertices inside C.

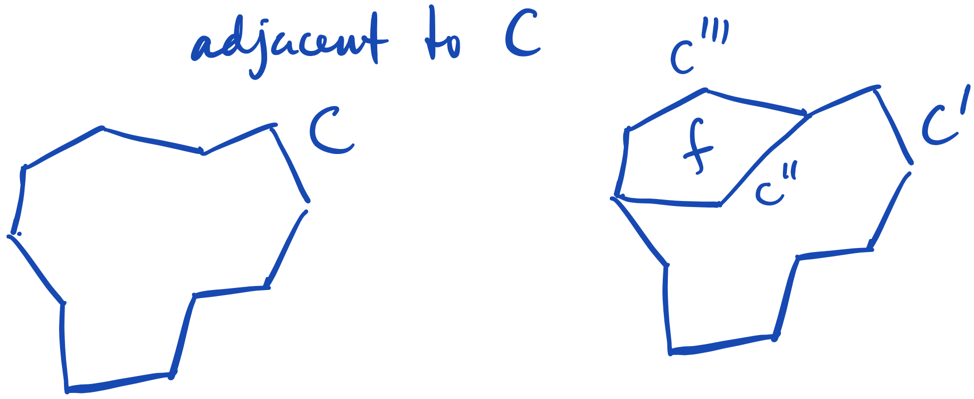
<u>Proof</u>. The induction in the number of faces:

- $K=1$, $(-1)^{n^K(C)} = -1$ by definition a K-orientation $\#(\text{vertices inside } C) = 0$, $\Rightarrow$ true

- Denote by $\overline{C} \subset \Gamma$ the cell subcomplex which consists of C and its interiour.

- Assume the statement is true for k and that C is enclosing $k+1$ faces and m vertices that a strictly insite C.

Choose a face $f \in F(\overline{C})$ which is adjacent to C



We have:

$n^k(C) = n^k(C''') + n^k(C') = n^k(C') +$

$n^k(C'') - n^k(C'') + n^k(C''') = n^k(C' \cup C'')$

$- n_+^k(\partial f \cap C'') + n^k(C''') = n_-^k(C' \cup C'') +$

$$+ n^K(\partial f) - n^K_+(\partial f \cap C'') - n^K(\partial f \cap C'') =$$

$$= n^K(C' \cup C'') + n^K(\partial f) - \ell =$$

$$= n^K(\partial(\overline{C} \setminus f)) + n^K(\partial f) - \ell$$

Cell subcomplex $\overline{C} \setminus f$ has k faces and therefore, by the induction assumption

$$n^K(\partial(\overline{C} \setminus f)) \bmod 2 = (m' + 1) \bmod 2$$

where m' = # vertices strictly inside $\partial(\overline{C}/f)$.

We have:

$$m = m' - \ell + 1$$

therefore

$$n^K(C) = (m'+1) + n^K(\partial f) - \ell =$$
$$= m + n^K(\partial f)$$

Because f is K-oriented, $n^K(\partial f) = 1 \bmod 2$ and therefore:

$$n^K(C) = (m+1) \bmod 2$$

∎

Corollary If C_i is a composition cycle

$$n^K(C_i) + 1 = 0 \mod 2$$

and therefore

$$\varepsilon^K(D)\,\varepsilon^K(D') = 1$$

for any pair D, D', $\Rightarrow \varepsilon^K(D)$ does not depend on D

This ends the proof of the proposition. ∎

The theorem immediately follows. ∎