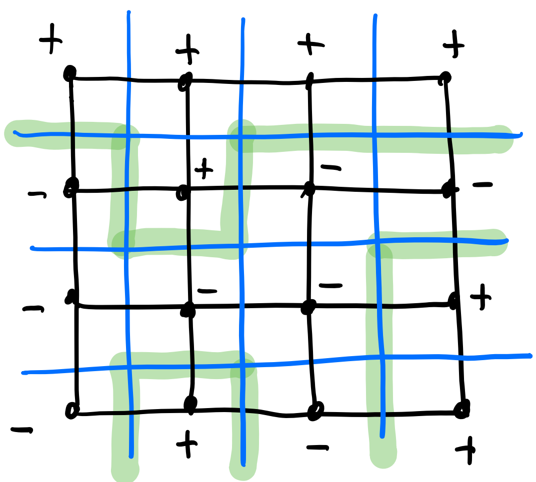
Mapping of the Ising model to a dimer model

Ising model: states are $s: V(\Gamma) \to \pm 1$

$\Gamma \subset$ square lattice (lattice domain connected simply connected)



"Domain walls" are paths on $\Gamma^\vee$ separating + and − regions

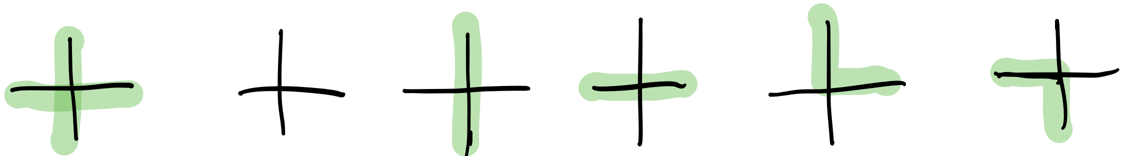
$$E(s) = -J_1 \sum_{\substack{\text{vertical} \\ \text{edges } (i,j)}} \sigma_i \sigma_j - J_2 \sum_{\substack{\text{horizontal} \\ \text{edges } (i,j)}} \sigma_i \sigma_j$$

Boltzmann distribution:

$$\mathrm{Prob}(s) = \frac{w(s)}{\sum_{s'} w(s')}, \quad w(s) = e^{-\beta E(s)}$$

$$\beta = \frac{1}{kT}$$

In terms of domain walls:



$e^{2\beta J_1 + 2\beta J_2}$ $\quad 1 \quad$ $e^{2\beta J_1}$ $\quad$ $e^{2\beta J_2}$ $\quad$ $e^{\beta J_1 + \beta J_2}$ $\quad$ $e^{\beta J_1 + \beta J_2}$

$e^{\beta J_1 + \beta J_2}$ $\quad$ $e^{\beta J_1 + \beta J_2}$

<u>Claim</u>

$$e^{-\beta E(\sigma)} = e^{-\beta J_1 |E_v| - \beta J_2 |E_h|} \, W(\text{domain walls})$$

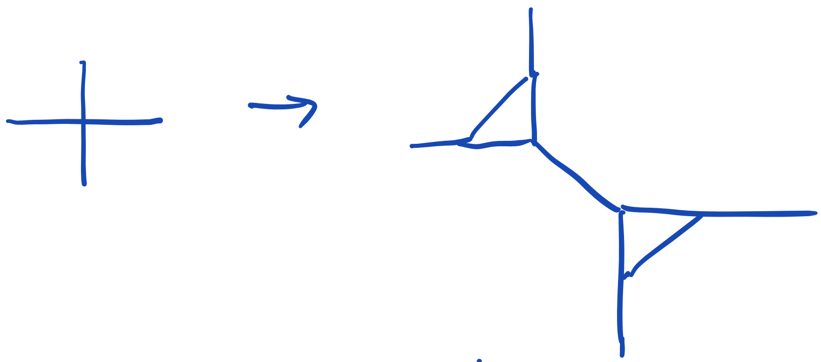
$$W(\text{domain walls}) = \prod_v w_v(\text{local domain wall near } v)$$

Now let us describe mapping to dimers

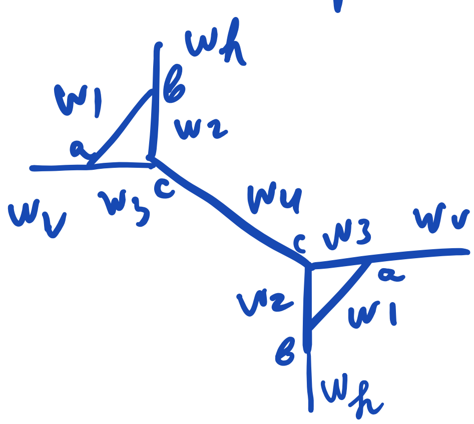
a) "Stretch" the dual lattice domain Γ^v:

$\bar{\ }^{v}$ $\quad \sim$

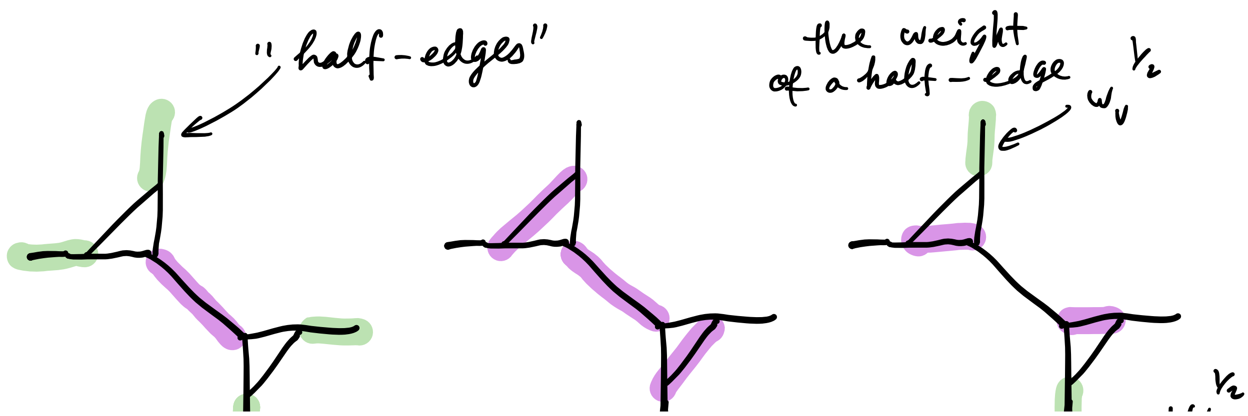
$\Gamma' \to \tilde{\Gamma}$



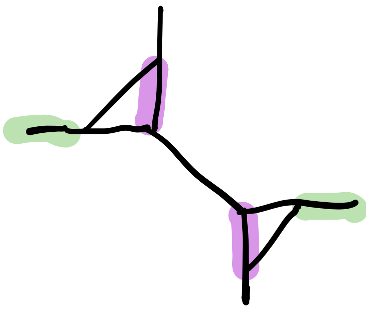
Assign dimer weights $w_1, w_2, w_3, w_4, w_v, w_h$ to edges of the graph $\tilde{\Gamma}$

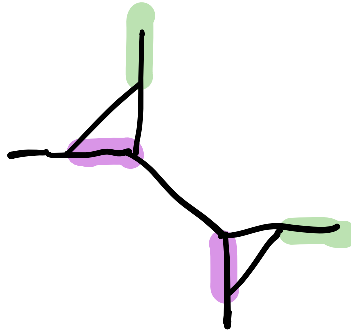


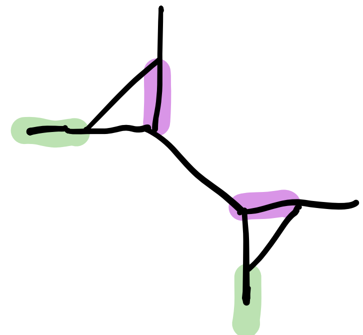
b) Extend Ising domain walls on $\Gamma^\vee$ to dimer configuration on $\tilde{\Gamma}$:



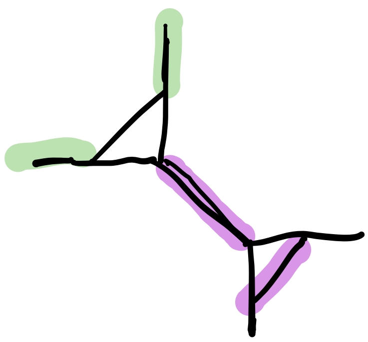
$w_h w_v w_4$ $w_1^2 w_4$ $w_v w_3^2$

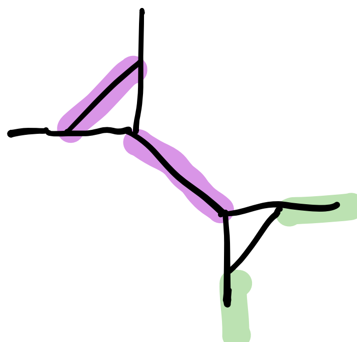






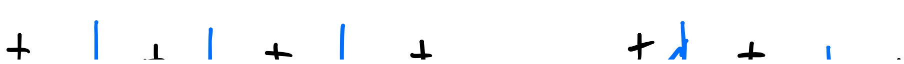
$w_h w_2^2$ $(w_h w_v)^{\frac{1}{2}} w_3 w_2$ $(w_h w_v)^{\frac{1}{2}} w_2 w_3$

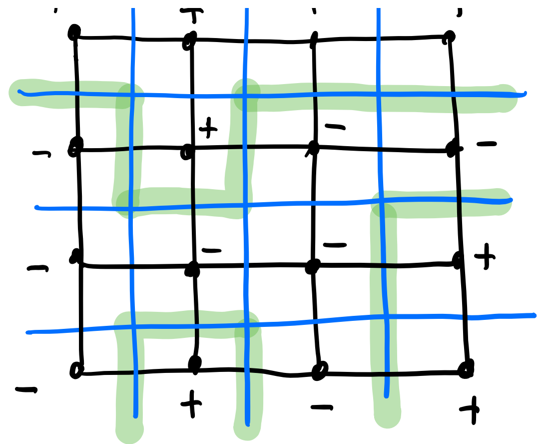


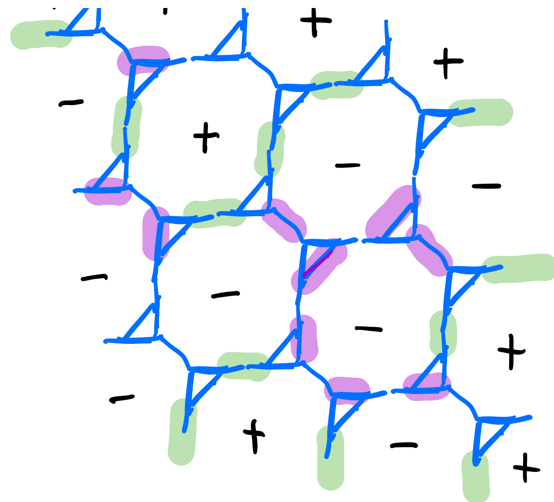


$(w_h w_v)^{\frac{1}{2}} w_4 w_1$ $(w_h w_v)^{\frac{1}{2}} w_1 w_4$

Clearly such extension is unique.
An example of such extension is given on Fig. 3:







Proposition When

$$w_4 = e^{2\beta J_1 + 2\beta J_2}, \quad w_3 = e^{\beta J_1}, \quad w_2 = e^{\beta J_2}$$

$$w_1 = e^{-\beta J_1 - \beta J_2}, \quad w_v = w_h = 1$$

the weight of a domain wall configuration on $\Gamma^\vee$ is equal to the weight of the corresponding dimer configuration on $\tilde{\Gamma}$.

Corollary

$$Z^{\text{Ising}}_{\Gamma} = Z^{\text{domain walls}}_{\Gamma^\vee} = Z^{\text{dimers}}_{\tilde{\Gamma}}$$

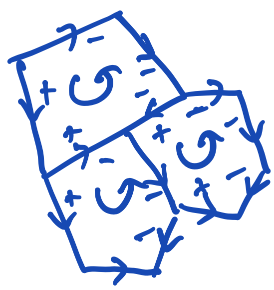
Note: $\tilde{\Gamma}$ is not bipartite, so no random surfaces here.

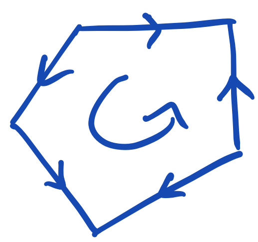
Kasteleyn solution

Computation of the partition function and of correlation functions in terms of Pfaffians.

$(\Gamma \subset \mathbb{R}^2)$

① Kasteleyn orientation of a plane graph.





$$\prod_{e \in \partial f} \varepsilon(e,f) = -1$$

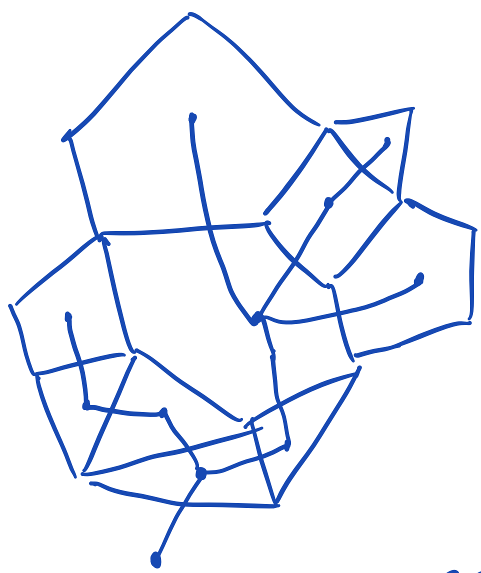
Proposition Kasteleyn orientations exist

Proof. h/w

Let us construct a Kasteleyn orientation.

. choose a spanning tree T for $\Gamma^\vee$

. orient edges of Γ that do not intersect T arbitrary



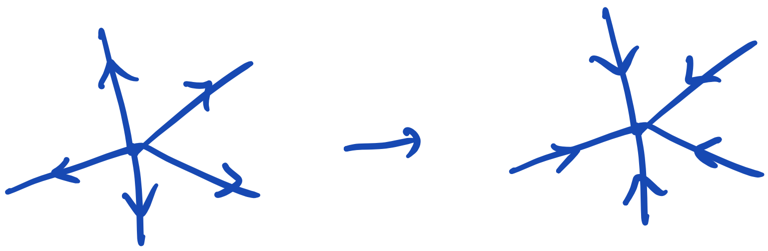
. orient edges on the boundary of faces corresponding to leaves of T such that they will be K-oriented.

. remove these faces and consider remaining cell complex with the corresponding spanning tree T'.

. continue this procedure.

The result will be a K-orientation of Γ. $\boxtimes$

Def. Two K-orientations K, K' are equivalent if K' can be obtained from K by a sequence of orientation reversing maps:



<u>Thm</u>. All K-orientations of a plane graph are equivalent.

<u>Proof</u>. h/w.

Let K_1, K_2 be two Kasteleyn orientations

define

$$\varepsilon_{K_1K_2}(e) = \begin{cases} 1, & K_1 \text{ orientation of } e \text{ coincide with } K_2\text{-orientation} \\ -1, & \text{otherwise} \end{cases}$$

Clearly
$$\prod_{e \in \partial f} \varepsilon_{K_1K_2}(e) = 1$$
for any face f.

Therefore $\varepsilon_{K_1K_2}$ defines a representation

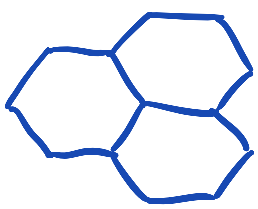
$\rho_{K_1K_2}: \pi_1(\Sigma) \to \mathbb{Z}_2$. Clearly $\varepsilon_{K_1K_2} = \varepsilon_{K_2K_1}$

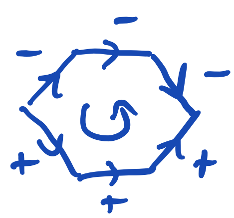
Lemma. • Each representation can be obtained in this way.

• If $K_1 \sim K_1'$, then $\rho_{K_1 K_2} = \rho_{K_1' K_2}$.

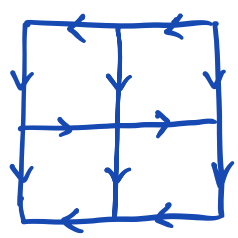
Proof. h/w.

Ex. 1)





2)



② Kasteleyn matrix:

Let $w(e) > 0$ be weights of edges and K be a Kasteleyn orientation

$$A^K_{ij} = \begin{cases} w(ij), & i \bullet\!\!\longrightarrow\!\!\bullet j \\ -w(ij), & i \bullet\!\!\longleftarrow\!\!\bullet j \\ 0, & \text{not connected} \end{cases}$$

<u>Theorem</u> (Kasteleyn)

$$\mathrm{Pf}(A^K) = \varepsilon^K \sum_{D \subset \Gamma} \prod_{e \in D} w(e)$$

where

$$\mathrm{Pf}(A) = \frac{1}{2^{n/2}(n/2)!} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n-1} A_{\sigma_i \sigma_{i+1}}$$

A : $n \times n$, skew symmetric, n is even

and

$$\varepsilon^K = (-1)^\sigma \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{n-1}\sigma_n}$$

does not depend on the permutation σ.

<u>Proof.</u> (see LN-38)

Digression on Grassman integrals and Pfaffians.

Consider $\mathbb{C}^n$,

$$\Lambda^{\circ}\mathbb{C}^n = \langle a_i \mid a_i a_j = -a_j a_i \,,\, i,j = 1 \dots n \rangle$$

Choose a basis in $\Lambda^n \mathbb{C}^n$, say $a_1 \dots a_n$

Any $f \in \Lambda^{\circ}\mathbb{C}^n$ can be written as

$$f = f_n\, a_1 \wedge \cdots \wedge a_n + \text{lower order terms}$$

<u>Def.</u> $$f_n = \int f \, da$$

<u>Remark</u>. Usially $da = da_n \cdots da_1$

and the integral is computed by the rules

$$\int a_i da_i = 1 \,, \quad \int da_i = 0 \,, \quad da_i a_j = - a_j da_i$$

<u>Lemma</u>. Let $A = (a_{ij})_1^n$ be a skew

symmetric matrix, then

$$\int \exp\left(\tfrac{1}{2}(a, Aa)\right) da = \mathrm{Pf}(A)$$

$$\exp(x) = \sum_{k\geq 0} \frac{1}{k!} x^k,$$

<u>h.w</u> Use Grassman integrals to prove

- $\mathrm{Pf}(A)^2 = \det(A)$

- If $\mathbb{C}^n = B \oplus W$, $\dim B = \dim W = \frac{n}{2}$

$$A = \begin{pmatrix} 0 & K^{BW} \\ K^{WB} & 0 \end{pmatrix}$$

$$\mathrm{Pf}(A) = \det_B(K^{BW} K^{WB}) =$$

$$= \det_W(K^{WB} K^{BW})$$

<u>Lemma</u> Let $A^t = -A$, $2n \times 2n$

(1) $$\frac{\partial^k \mathrm{Pf}(A)}{\partial a_{i_1 j_1} \cdots \partial a_{i_k j_k}} = (-1)^{\sigma(I)} \mathrm{Pf}(A_I)$$

where A_I = matrix A with colouns and rows $i_1 j_1 \ldots i_k j_k$ being removed

and $\sigma : (1, 2, \ldots, 2n) \mapsto (i_1 j_1 \ldots i_k j_k \, 1 \ldots \hat{i_1} \, \hat{j_1} \ldots \hat{i_k} \, \hat{j_k} \ldots 2n)$

(2) $$(-1)^{\sigma(I)} \mathrm{Pf}(A_I) = (-1)^k \mathrm{Pf}(A) \, \mathrm{Pf}\big((A^{-1})_{ab}\big)_{a,b \in I}$$

<u>Thm</u> when $(i_a j_a) \neq (i_b j_b)$

$$\langle \sigma_{i_1 j_1} \cdots \sigma_{i_k j_k} \rangle = (-1)^k a^k_{i_1 j_1} \cdots a^k_{i_k j_k} \cdot$$

$$\cdot \; \mathrm{Pf}\big(((A^k)^{-1})_{ab}\big)_{a,b \in \{i_1 \ldots i_k j_1 \ldots j_k\}}$$

<u>Proof.</u>

$$\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle = \frac{1}{Z} w(e_1) \cdots w(e_k) \frac{\partial^k Z}{\partial w(e_1) \cdots \partial w(e_k)} =$$

$$\partial^k \; \cdots \; \mathrm{Pf}(A^k)$$

$$= \frac{1}{Z} w(e_1)\cdots w(e_k) \frac{\text{[illegible]}}{\partial w(e_1)\cdots\partial w(e_k)} =$$

$$= \frac{1}{Pf(A)} a^K_{i_1 j_1} \cdots a^K_{i_k j_k} \frac{\partial^k Pf(A^k)}{\partial a^K_{i_1 j_1} \cdots \partial a^K_{i_k j_k}},$$

we can assume $i_1 < j_1 \ldots i_k < j_k$

Then use Lemma.

In particular:

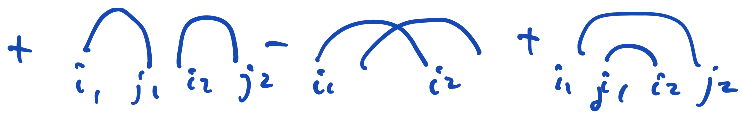
$$\langle \sigma_{ij} \rangle = -a_{ij} (A^K)^{-1}_{ij}$$

For two point correlation function:

$$\langle \sigma_{i_1 j_1} \sigma_{i_2 j_2} \rangle = a_{i_1 j_1} a_{i_2 j_2} \Big((A^{-1})_{i_1 j_1} (A^{-1})_{i_2 j_2} +$$

$$+ (A^{-1})_{i_1 j_2} (A^{-1})_{i_2 j_1} - (A^{-1})_{i_1 i_2} (A^{-1})_{j_1 j_2} \Big)$$

graphically these three terms are:



+
i1 j1 i2 j2
−
i1 i2
+
i1 j1 i2 j2