Lect 2

Dimers on bipartite lattice domains and height functions (discrete surfaces)

1) $\Gamma \subset \mathcal{L} \subset \mathbb{R}^2$ lattice domain
(connected simply connected)

Chain complex:

$$C_\bullet(\Gamma) = C_0(\Gamma) \oplus C_1(\Gamma) \oplus C_2(\Gamma)$$

$$C_0(\Gamma) = \{ \sum_v c_v v \,,\ c_v \in \mathbb{R} \}$$

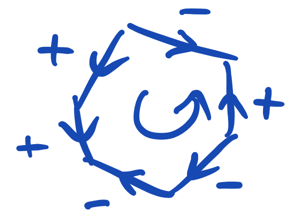
$$C_1(\Gamma) = \{ \sum_e c_e e \,,\ c_e \in \mathbb{R} \}$$

$$C_2(\Gamma) = \{ \sum_f c_f f \,,\ c_f \in \mathbb{R} \}$$

Boundary operation:

$$C_2(\Gamma) \xrightarrow{\partial} C_1(\Gamma) \xrightarrow{\partial} C_0(\Gamma)$$

$$\partial f = \sum_{e \in \partial f} (-1)^{(e,f)} e ,$$



represents the boundary of f,

$$\partial e = e_+ - e_- , \quad e_+ \xrightarrow{e} e_-$$

represents the boundary of e,

$$\partial v = 0$$

Note: defined over $\mathbb{Z}$.

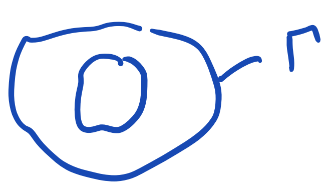
<u>Thm</u>. $\partial^2 = 0$

<u>Proof</u>. h.w.

$$Z_1 = \{ c \in C_1 , \partial c = 0 \} = \ker(\partial) \subset C_1$$

represent closed cycles (no boundary points)

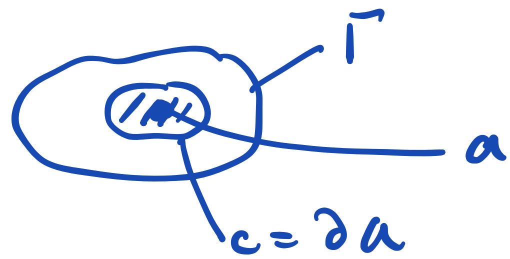
combinations of



$B_1 = \{ c \in C_1,\ c = \partial a,\ a \in C_2 \} = \partial(C_2) \subset C_1$

represent boundaries of discs

combinations of



Homology spaces (1st homologies of the cell complex $C_\bullet(\Gamma)$)

$H_1 = Z_1 / B_1,$

(combinations of equivalence classes of closed paths on Γ which are not boundaries of discs (noncontractible))

<u>Thm</u>. For connected, simply connected

$\Gamma \subset \mathcal{L} \subset \mathbb{R}^2, \qquad H_1 = \{0\}$

<u>Proof</u>. h.w.

Note: not true if Γ is not simply connected

$\Gamma \subset \mathcal{L} \subset \mathbb{R}^2$

Corollary. If $c \in C_1$ and $\partial c = 0$ $\exists\, h \in C_2$, s.t. $c = \partial h$.

Note that

$$C_2 = \left\{ \sum_f c_f f \,,\ f \text{ is a 2-cell of } \Gamma \right\}$$

$$= \left\{ \sum_{v \in \Gamma^\vee} h_v v \,,\ v \text{ is a 0-cell in } \Gamma^\vee \right\}$$

2) Assume Γ is a bipartite graph. Because it is bipartite, we can orient it as $\bullet \!\longrightarrow\! \circ$.

Let $D \subset \Gamma$ be a dimer configuration

Define
$$\hat{D} = \sum_e e \in C_1(\Gamma)$$

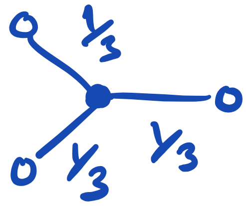
$$\partial \hat{D} = \sum_{b \in B} b - \sum_{w \in W} w = B - W$$

Let $\omega \in C_1(\Gamma)$ be such that we also have

$$\partial\omega = B - W$$

<u>Examples:</u> 1) $\omega = \hat{D}_\circ$ for some dimer configuration

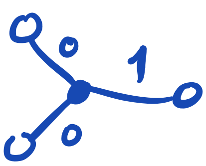
2) hexagonal lattice :



$$\partial\omega = \sum_{e \in \Gamma} \frac{1}{3}(b - w) = B - W$$

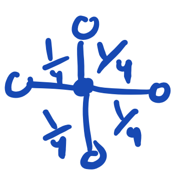
<u>h.w. check it</u>

3) <u>hexagonal lattice:</u>



$$, \quad \partial\omega = \sum_{\substack{\text{horizonal} \\ \text{edges}}} \partial e = B - W$$

4) <u>Square lattice</u> :



$$\partial\omega = B - W$$

h.w.

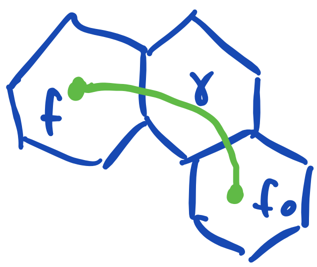
We have

$$\partial(D-\omega)=0$$

Therefore $\exists h$ s.t. $D-\omega=\partial h_{D,\omega}$

Explicit construction of $h_{D,\omega}$:

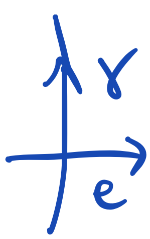
Fix f_0, a face on Γ and a path γ connecting f_0 and f



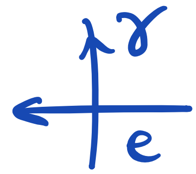
and the value $h_{D,\omega}(f_0)$.

Then

$$h_{D,\omega}(f)=\sum_{e\cap\gamma\neq\phi}\varepsilon(e,\gamma)\big(\sigma_D(e)-\omega(e)\big)+$$
$$+\,h_{D,\omega}(f_0)$$



$\varepsilon(e,\gamma)=+1$,



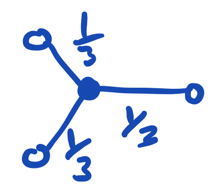
$\varepsilon(e,\gamma)=-1$

Thm. • $h_{D,\omega}(f)$ does not depend on γ

• $\partial h_{D,\omega} = \hat{D} - \omega$

Proof. h.w.

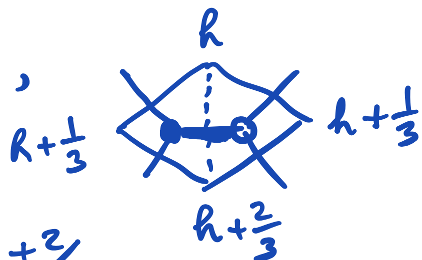
Examples:

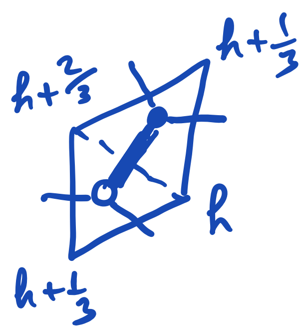
1) hexagonal lattice H, ω: 

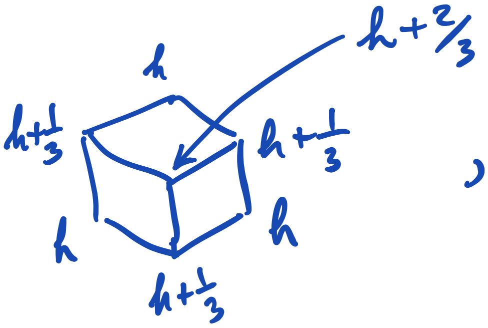
Faces of the hexagonal lattice H =
= vertices of the dual triangular lattice T

Dimers on H = rombi tilings of T

$\partial h = \hat{D} - \omega$,







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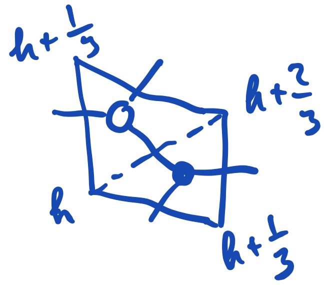
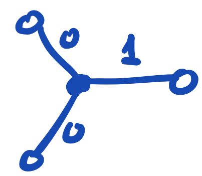
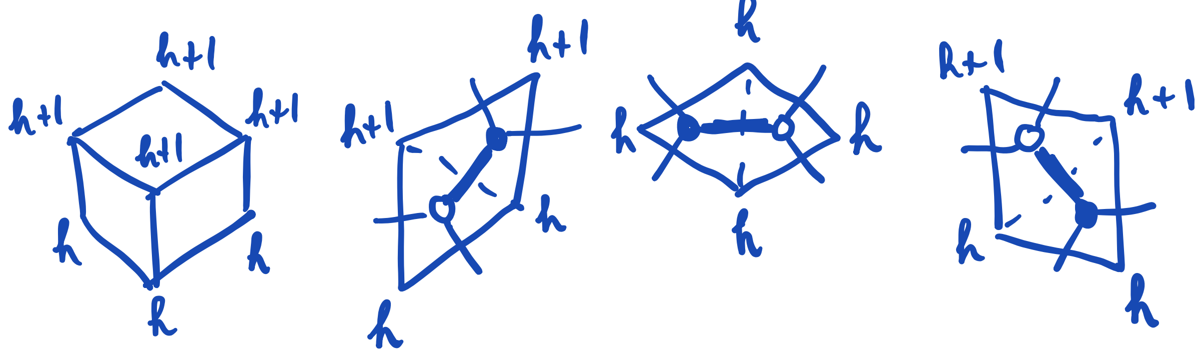


h = the distance along $(1,1,1)$ in the 3D space

2) hexagonal lattice H, ω: 

$\partial h = \hat{D} - \omega$,



h = the distance to the "floor".

3) $\omega = \hat{D}_0$, D_0 - fixed dimer configuration

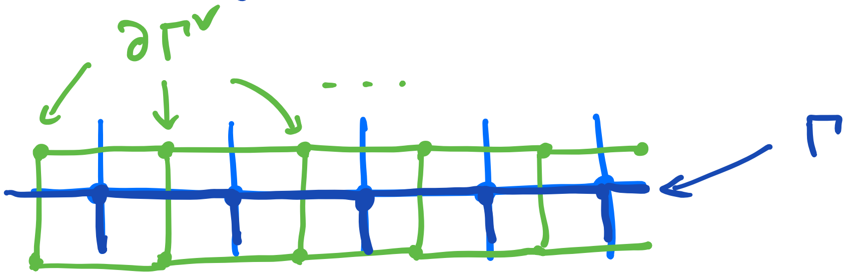
$$\partial h_{D,D_0} = \hat{D} - \hat{D}_0$$

Some properties of height functions:

- $h_{D,\omega_1} - h_{D,\omega_2} = h_{\omega_1,\omega_2}$

$\bullet \quad h_{D_1,\omega} - h_{D_2,\omega} = h_{D_1,D_2}$

Boundary values of height functions



$\partial\Gamma^\vee$ - boundary vertices of $\Gamma^\vee$

Boundary faces of Γ

Thm For $D \subset \Gamma$, $h_{D,\omega}\big|_{\partial\Gamma^\vee}$ does not depend on D.

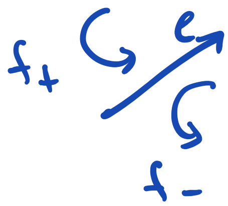
Proof. $\partial h_{D,\omega} = \hat{D} - \omega$

$$h_{D,\omega} = \sum_f h_{D,\omega}(f)\, f$$

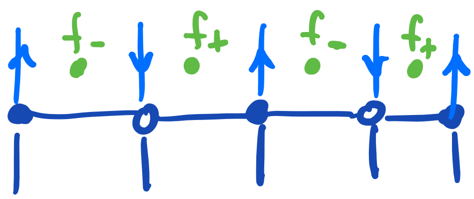
$$\partial h_{D,\omega} = \sum_f h_{D,\omega}(f) \sum_{e \in \partial f} \varepsilon(e,f)\, e$$

$$= \sum_e \Big(\sum_{f:\, e \in \partial f} \varepsilon(e,f)\, h_{D,\omega}(f) \Big)\, e$$

$$(\partial h_{D,\omega})(e) = h_{D,\omega}(f_+) - h_{D,\omega}(f_-)$$



$$h_{D,\omega}(f_+) - h_{D,\omega}(f_-)$$
$$= \sigma_e(D) - \omega(e)$$



Because dimers can not be on blue edges

$\Rightarrow$ along the boundary

$$h_{D,\omega}(f_\varepsilon) - h_{D,\omega}(f_{-\varepsilon}) = -\omega(e_\varepsilon)$$

does not depend on D. $\blacksquare$

The space of height functions

$$\mathcal{H}_{\Gamma,\omega} = \{ h_{D,\omega} \}$$

partially ordered set, $\exists\, h_{min}, h_{max}$

Because $h_{D,\omega_1} - h_{D,\omega_2} = h_{\omega_1,\omega_2}$

$$\mathcal{H}_{\Gamma,\omega_1} = \mathcal{H}_{\Gamma,\omega_2} + \{h_{\omega_1,\omega_2}\}$$

Boltzmann weights in terms of height functions

We have bijections:

Dimers on Γ $\longleftrightarrow$ height fncns on $\Gamma^\vee$

$\nwarrow\!\!\searrow$ $\quad$ $\swarrow\!\!\nearrow$

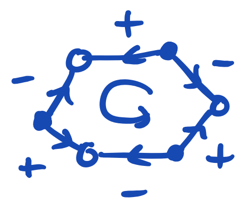
Tilings of $\Gamma^\vee$ by double tiles.

From now on we focus on dimers $\leftrightarrow$ h. fncns bijection.

Thm. Let $h_{D,w}$ be a height function

$$\prod_{e \in D} w(e) = \prod_{e \in \Gamma} w(e)^{\omega(e)} \cdot \prod_{f} q_f^{\,h_{D,w}(f)}$$

$$q_f = \prod_{e \in \partial f} w(e)^{\varepsilon(e,f)}$$



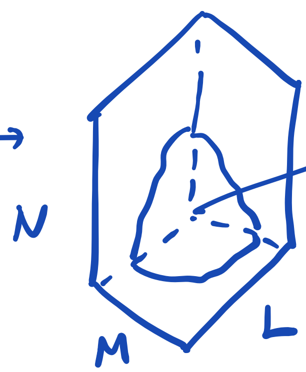
Proof. h.w.

Corollary

$$\mathrm{Prob}(D) = \frac{\prod_{e\in D} w(e)}{\sum_{D'\subset\Gamma}\prod_{e\in D'} w(e)} = \frac{\prod_f q_f^{h_{D,w}(f)}}{\sum_{h\in\mathcal{H}_{\Gamma,w}}\prod_f q_f^{h(f)}}$$

Example:

hexagon on $T = H^\vee$



If $q_f = q$, $\forall f$

(h.w. compute $w(e)$ corresponding to these q_f)

$$\mathrm{Prob}(\pi) = \frac{1}{Z} q^{|\pi|},$$

$|\pi|$ = the number of cubes in the pile π corresponding to D

An interesting particular case:

$N = \infty$, $q < 1$

$$\mathrm{Prob}(\pi) = \frac{1}{Z} q^{|\pi|},$$

$$Z = \sum_{\pi} q^{|\pi|} =$$

How to compute Z_{dimer}
and local correlation functions?