Lect 4

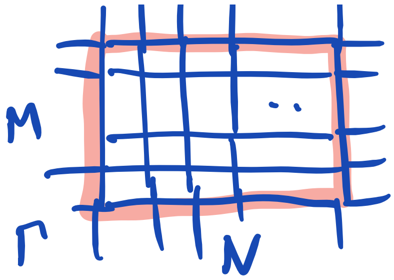
Kast. orient. → Kast. sign fcn

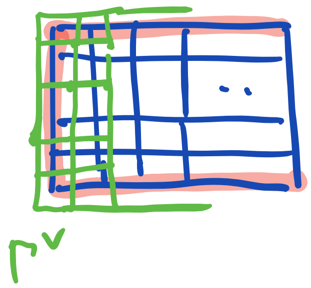
1. Kasteleyn orientations on square lattice domains
2. Perkus-Kasteleyn operators → complex valued
3. The partition function ~~and correlation functions~~ for rectangular domains
4. The thermodynamic limit
5. The limit shape phenomenon and the variational principle
6. Aztec dimond with 2-periodic weights. Ordered phase.

1 Two special regions on a square lattice

1. A rectangular domain of size $N \times M$

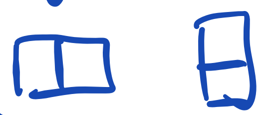
Everything can be calculated explicitly





by 1 "step bigger than Γ"

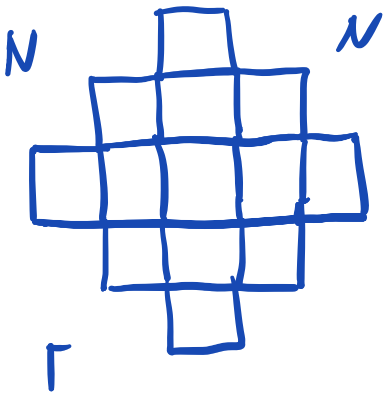
dimers on Γ $\longleftrightarrow$ tilings of $\Gamma^\vee$ by dominos

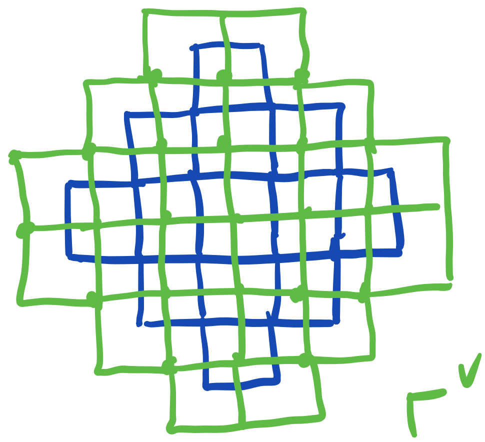


↑

height fn = discrete surface →

2. An Aztec diamond of size $N \times N$





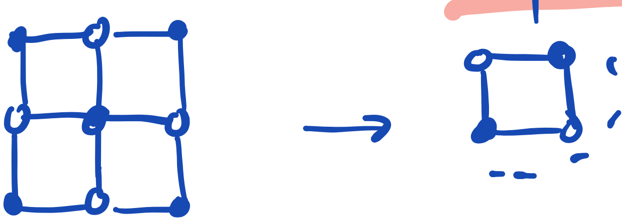
Dimers Γ

↓

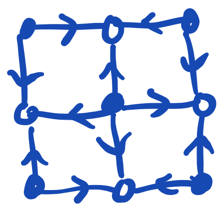
$N=3$

It has an interesting limit shape.

② Square lattice is bipartite

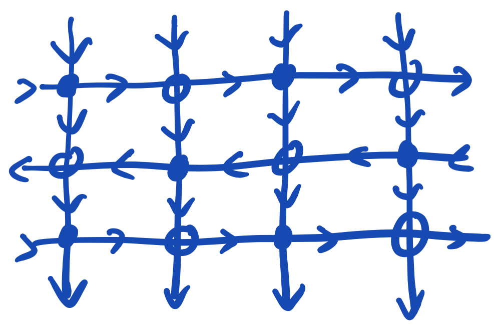


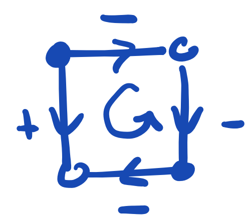
This gives a bipartite orientation of Γ



black $\to$ white

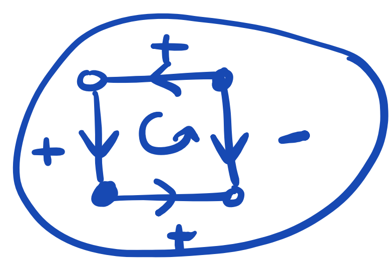
③ Kasteleyn orientation for the square grid





$\prod$ signs $= -1$

same :



OK

$\prod$ signs $= -1$

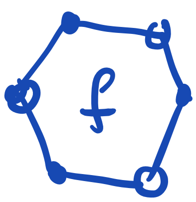
Introduce the Kast. sign fncn.

$\varepsilon : E(\Gamma) \to \pm$

Here we use the bipartite str.

$$\varepsilon_K(e) = \begin{cases} +1, & \text{K-orient of } e \text{ agrees with the bipartite one} \\ -1, & \text{otherwise} \end{cases}$$

Kasteleyn condition for the sign function $\varepsilon_K(e)$ is:



$$\prod_{e\in\partial f} \varepsilon_K(e) = -1$$ for any face f of Γ

from $\prod_{e\in\partial f} \varepsilon_K(e,f) = -1$

the relative orient, as above

and

from $\prod_{e\in\partial f} \varepsilon_B(e,f) = 1$

for the bipartite orient.

The Kasteleyn matrix is:

$$A_{..}\, b = \sum \varepsilon^K(b,w)\, w(b,w)\, w$$

$$A_K w = -\sum_{w-b} \varepsilon^K(w,b)\, w(w,b)\, b \quad \Bigg\}\ \text{no bipartite str. in } \varepsilon^K(v,v')$$

This the original formula

$$\varepsilon^K(v,u) = \begin{cases} +1, & v \to u \\ -1, & u \to v \end{cases}$$

Weight $w(e)$ do not depend on the orientation and

$$\varepsilon_K(b,w) = \varepsilon_K(e), \quad \varepsilon(w,b) = -\varepsilon_K(e)$$

<u>For a bipartite graph:</u> $V = C_0(\Gamma)$ - v. space spanned by vertices

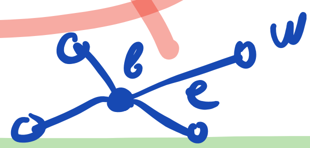
$$A_K = \begin{pmatrix} 0 & \tilde{A}_K \\ -\tilde{A}_K^t & 0 \end{pmatrix}$$

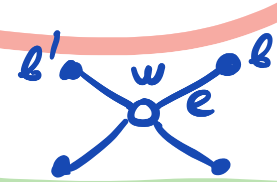
$\tilde{A}_K : B \to W$

$V = B \oplus W$

spanned by black v. ; w-vertices

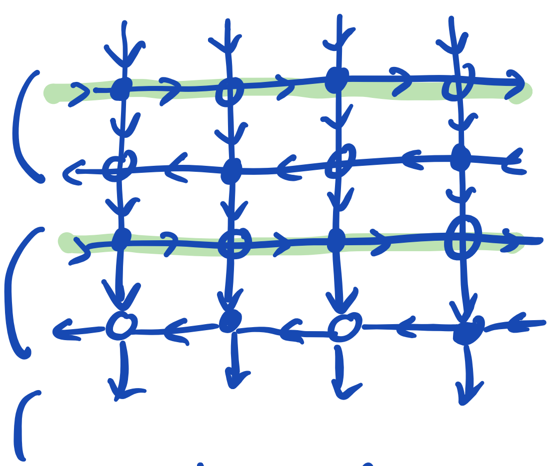
$$\tilde{A}_K b = \sum_{e,\ b \in \partial e} \varepsilon^K(e)\, w, \qquad \tilde{A}^t w = \sum_{e,\ w \in \partial e} \varepsilon^K(e)\, b$$



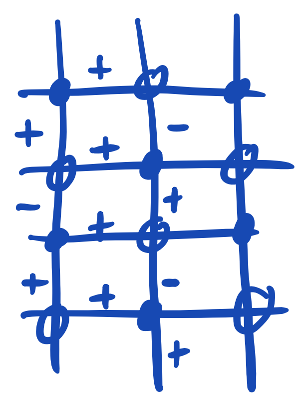


④ Now let us compute the Pfaffian of A_K for a rectangular square domain with the K-orientation from above.

We know: $Z^{dimer}_{\Gamma} = \sum_{D} \prod_{e \in D} w(e) = |\,\mathrm{Pf}(A^K)\,|$



The Kasteleyn orientation



corresponding sign fnctn

A^K is a difference operator transl. invariant in the horizontal direction and in vertical direction it has step 2 (I can use the discrete Fourier transform to find the spectrum)

Lemma Let $s : V(\Gamma) \to \mathbb{C}$. The transformation

$$w(e) \mapsto s(e_+)\, w(e)\, s(e_-)$$

does not change the probability distribution of the dimer model.

Proof.

$$\mathrm{Prob}(D) = \frac{\prod_{e \in D} w(e)}{\sum_{D'} \prod_{e \in D'} w(e)}$$

$$w'(e) = s(e_+)\, w(e)\, s(e_-)$$

inv. $e_+ \Leftrightarrow e_-$

(gauge transformation)

$$\mathrm{Prob}'(D) = \frac{\prod_{e\in D} s(e_+)\, w(e)\, s(e_-)}{\sum_{D'} \prod_{e\in D'} s(e_+)\, w'(e)\, s(e_-)} =$$

D covers all vertices each of them once

$$= \frac{\left(\prod_{v\in\Gamma} s(v)\right) \prod_{e\in D} w(e)}{\sum_{D'} \left(\prod_{v\in\Gamma} s(v)\right) \prod_{e\in D'} w(e)} = \mathrm{Prob}(D).$$

■

$$A^K_{vv'} = \varepsilon^K(v,v')\, w(e_{vv'})$$

$$A'^K_{vv'} = \varepsilon^K(v,v')\, s(v)\, s(v')\, w(e_{vv'})$$

$$A'^K = S A^K S$$

$S v = s(v)\, v$ diagonal matrix in V

① when $s(v) > 0$, these transformations change weight, do not change $\mathrm{Prob}(D)$.

Weight / such gauge transf. =

= "effective" parameters of $\mathrm{Prob}(D)$.

② $s(v) \in \mathbb{R}$, $s(v) = \sigma(v)\,|s(v)|$

$$A'^{K}_{vv'} = \varepsilon^{K}(v,v')\,\sigma(v)\,\sigma(v') \cdot$$

$$\cdot\ |s(v)|\, w(e_{vv'})\, |s(v')|$$

Claim (h.w.)

$$\varepsilon^{K}(v,v')\,\sigma(v)\,\sigma(v') = \varepsilon^{K'}(v,v')$$

where K' is equivalent to K through change of orientation at vertices with $\sigma(v) = -1$.

$$A'^{K} = |S| \, A^{K'} \, |S|$$

③ $s(v) \in \mathbb{C}$, $|s(v)|$ change w/ $e_{v,v'}$

$s(v)/|s(v)|$ they change the Kast signs to compl. numbers

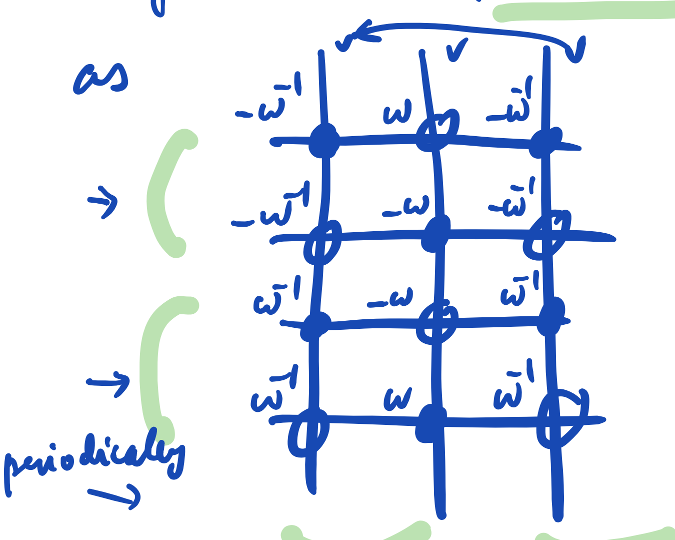
. R. Kenyon (discrete Dirac & Laplace operators)

Discrete geometry related to this

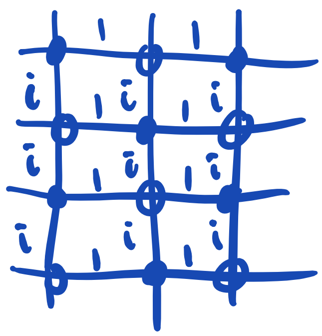
Let $\omega = e^{-i\frac{\pi}{4}}$, $\omega^2 = -i$.

Define the function $s : V(\Gamma) \to \mathbb{C}$

as



Thm (Percus) After this transformation the complex numbers $\varepsilon^K_{new}(e)$ in the Kasteleyn matrix are:



$\varepsilon^K_{new}(e)$

$$\varepsilon^K_{new}(e) = s(e_+)\,\varepsilon^K(e)\,s(e_-)$$

as above

$\pm$

Proof (h.w.)

Assume $w(e)=1$ (uniform distribution)

Now the complex valued Kast matrix:

skew symm

$$A^K = \begin{pmatrix} 0 & \tilde{A} \\ -\tilde{A}^t & 0 \end{pmatrix}, \quad \tilde{A} : B \to W, \quad \tilde{A}^t : W \to B$$

$$\boxed{\tilde{A}\, b_{n,m} = w_{n+1,m} + w_{n-1,m} + i\left(w_{n,m+1} + w_{n,m-1}\right)}$$

$b_{n,m}$: $n = m \mod 2$

$w_{n,m}$: $n = m+1 \mod 2$

$|Pf(A^K)| = |\det(\tilde{A})|$ (lect. 3)

$\exists \ K = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^t & 0 \end{pmatrix} \leftarrow$ symmetric

$\det\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = (-1)^{\frac{n}{2}} \det(A)\det(B)$

$$\boxed{|Det(K)| = |\det(\tilde{A})^2|}$$

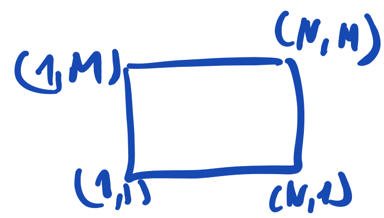
Thus $|Pf(A^K)| = |Det(K)|^{\frac{1}{2}}$

(1) $K v_{n,m} = v_{n+1,m} + v_{n-1,m} + i\left(v_{n,m+1} + v_{n,m-1}\right)$

Let us find the spectrum of K

(2) b.c. $v_{0,m} = v_{n,0} = 0$

$v_{N+1,m} = 0 \quad v_{n,M+1} = 0$



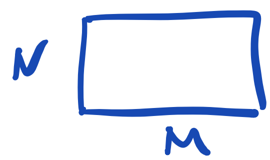
Thm. Vectors

$$y_{k,l} = \sum_{n,m=1}^{N,M} v_{n,m} \sin\left(\frac{\pi k n}{N+1}\right) \sin\left(\frac{\pi m l}{M+1}\right)$$

where $k = 1,\dots,N$, $l = 1,\dots,M$ form a complete set of eigenvectors with eigenvalues

$$\boxed{\lambda_{k,l} = 2\left(\cos\left(\frac{\pi k}{N+1}\right) + i\cos\left(\frac{\pi l}{M+1}\right)\right)}$$

Proof. h.w.



~~$NM = \#(V)$ is odd~~ → no perfect matchings

Note, for an $N\times M$ region to have a single dimer configuration at least one of the numbers N, M should be even.

Corollary 1.

$$\boxed{Det(K) = \prod_{k=1}^{N}\prod_{l=1}^{M} 2\left(\cos\left(\frac{\pi k}{N+1}\right) + i\cos\left(\frac{\pi l}{M+1}\right)\right)}$$

Corollary 2

$$\boxed{|Pf(A^k)| = \prod_{k=1}^{N}\prod_{l=1}^{N} 2\left|\cos\left(\frac{\pi k}{N+1}\right) + i\cos\left(\frac{\pi l}{M+1}\right)\right|^{\frac{1}{2}}}$$

Corollary 3 The number of tilings of an $(N+1)\times(M+1)$ square region by dominoes is

\# of dimer

$$\boxed{\#(\text{tilings}) = |\mathrm{Pf}(A^k)|}$$ $\approx$ config. on Γ

$= Z_{\Gamma}^{\text{dimer}}$ (uniform)

⑤ Thermodynamic limit (uniform weights)

For uniform weights $w(e) = 1$

$$Z_{NM}^{\text{dimers}} = \prod_{k=1}^{N}\prod_{\ell=1}^{M} \left|2\left(\cos\left(\frac{\pi k}{N+1}\right) + i\cos\left(\frac{\pi \ell}{M+1}\right)\right)\right|^{\frac{1}{2}}$$

$$= \exp\left(\frac{1}{2}\sum_{k=1}^{N}\sum_{\ell=1}^{M} \ln\left(2\left|\cos\left(\frac{\pi k}{N+1}\right) + i\cos\left(\frac{\pi \ell}{M+1}\right)\right|\right)\right)$$

Riemann sum

as $N, M \to \infty$

$$\left\{\frac{\pi k}{N+1}\right\}_{k=1}^{N} \xrightarrow[N\to\infty]{} \underset{\theta}{(0,\pi)}$$

$$\sum_{k=1}^{N}\sum_{\ell=1}^{M} \cdots \to \frac{NM}{2} \boxed{\int_0^{\pi}\int_0^{\pi} \ln\left|2\left|\cos(\theta) + i\cos(\varphi)\right|\right| d\varphi\, d\theta}$$

the free energy per site

$$Z_{NM}^{\text{dimer}} = \exp\left(NMG + o(NM)\right) = \exp\left(\mathrm{Vol}(\Gamma)G + \cdots\right)$$

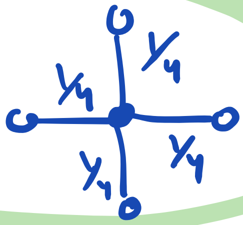
$$\boxed{G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots,\ \text{Catalan's const.}}$$

⑥ Random surfaces in the thermodynamic limit.

(uniform weights)

Dimers on Γ ⟷ Tilings of $\Gamma^\vee$ by double tiles

discrete surfaces defined by height fncns

ω: 

particular choice

Recall that a height function is a function on $\Gamma^\vee$ with $\hat{D} = \sum_{e \in D} e \in C_1(\Gamma)$

$$\partial h_D = \hat{D} - \omega \qquad \left(\begin{array}{l} \partial \hat{D} = B - W \\ \partial \omega = B - W \end{array} \right) \text{ assume the bipartite orient.}$$

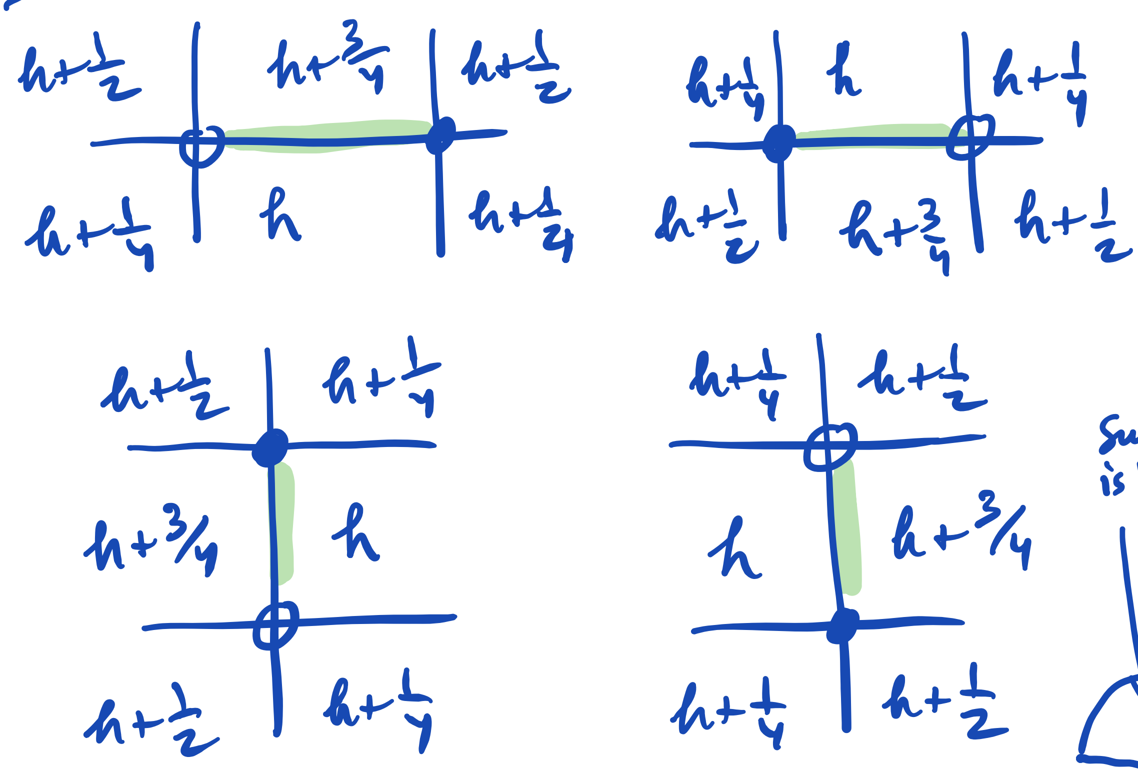
$\partial(\hat{D} - \omega) = 0$ since $H_1(\Gamma) = 0$

$\omega \in C_1(\Gamma)$ we choose it

$$\partial h = \hat{D} - \omega$$

for a square lattice

the height function changes around a dimer edge as:



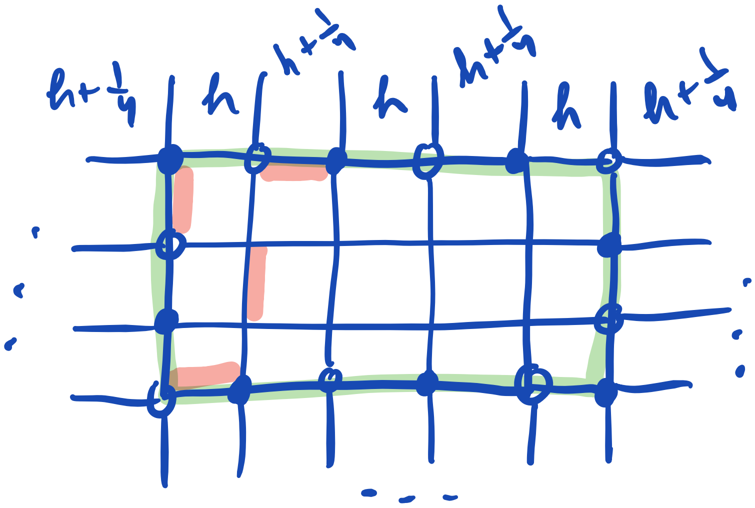
surface is the graph of $h_{n,m}$

Consider $N \times M$ rectangular region

NM should be even.

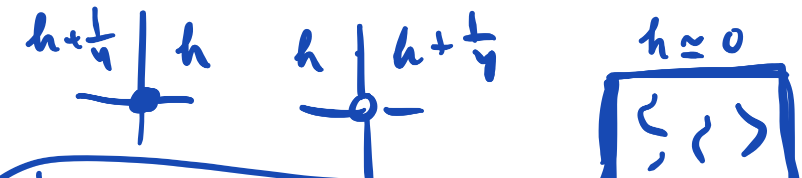
$$\left(\text{Dimers on } \Gamma\right) \Longleftrightarrow \left(\begin{array}{c}\text{height functions (a discrete surface)} \\ \text{on faces of } \Gamma \\ (\text{on vertices of } \Gamma^\vee)\end{array}\right)$$

①



— dimers on Γ

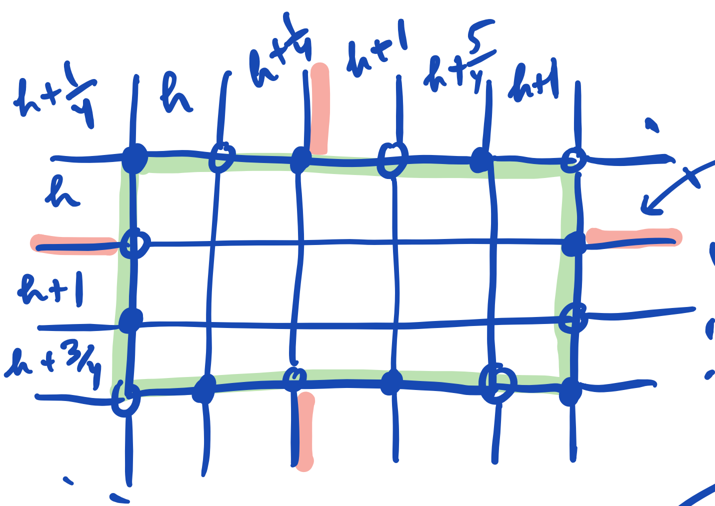
it changes periodically on $\partial\Gamma$ approximately constant.



$h|_{\partial\Gamma}$ is compl. det. by Γ & ω

This is what we studied

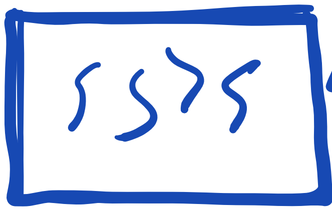
② Dimers are placed on adjacent to Γ vertices



boundary dimers.

Let us fix some config. of such dimers. $D|_{\partial\Gamma}$

The boundary h.f. is completely determined by Γ, ω, $D|_{\partial\Gamma}$



φ on the boundary

boundary h. fncn.

Assume that as $N, M \to \infty$, $N = \frac{L_1}{\varepsilon}$, $M = \frac{L_2}{\varepsilon}$, $\varepsilon \to 0$, boundary dimer configuration induce boundary value of the height function

data

$$\Downarrow \quad h_{n,m} \to \frac{1}{\varepsilon}\varphi(x,y), \quad n = \frac{x}{\varepsilon}, \quad m = \frac{y}{\varepsilon}$$

$n, m \in \partial\Gamma$

Thm (Cohn, Kenyon, Propp)

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln\left(Z^{\text{dimer}}_{D_0}\right) =$$

↑ boundary dimer config.

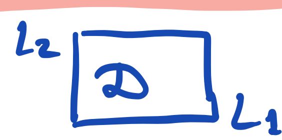
$$= \int_0^{L_1}\int_0^{L_2} \sigma(\vec{\nabla} h_0)\, dx\, dy$$

where h_0 is the minimizer of

$$S[h] = \int_0^{L_1}\int_0^{L_2} \sigma(\vec{\nabla} h)\, dx\, dy$$

+ Lipshitz condition

with $h|_{\partial D} = \varphi$



in $\mathbb{R}^2$

The function $\sigma(s,t)$ is the Legendre transform of

Rhonkin function

$$f(H,V) = \iint\limits_{|z|=e^H,\ |w|=e^V} \ln\left(\left|P\left(z+z^{-1}+i(w+w^{-1})\right)\right|\right) \frac{dz}{z}\frac{dw}{w}$$

$|P(z,w)|$

Proof. ... complicated ...

$\varepsilon \to 0$

$$Z^{\text{dimer}}_{\Gamma, D_0} \simeq \exp\left(\frac{\mathrm{Vol}(D)}{\varepsilon^2}\,{}^{,,L_1L_2}S[h_0] + \text{lower order terms}\right)$$

Remark. Related problem:

Γ, Δ_Γ = the adjacency matrix of Γ

discrete difference operator

$$\det(\Delta_\Gamma) \longrightarrow \exp\left(\frac{\mathrm{Vol}(D)}{\varepsilon^2} S + \cdots\right)$$

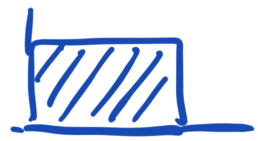
More general:

active research

$$\det(D) \xrightarrow{|\Gamma| \to \infty} \;?$$

difference operator on $\Gamma \subset \mathbb{Z}^2$

Remark. If $\varphi = 0$ (no dimers are sticking out of Γ) $h_0 = 0$,



h_0 = the limit shape, deterministic

As $\varepsilon \to 0$

$$h \approx \frac{1}{\varepsilon} h_0 + \frac{1}{\sqrt{\varepsilon}} \varphi$$

random, fluctuations

fluctuate at a smaller scale.

random

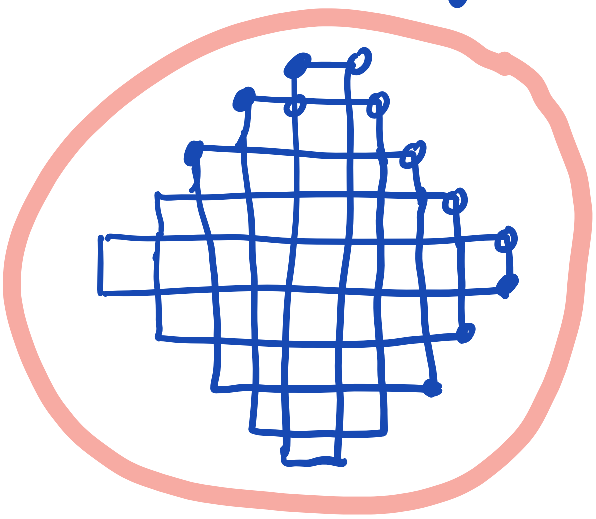
$$\varepsilon h \underset{\varepsilon \to 0}{\simeq} h_0 + \sqrt{\varepsilon}\, \varphi$$

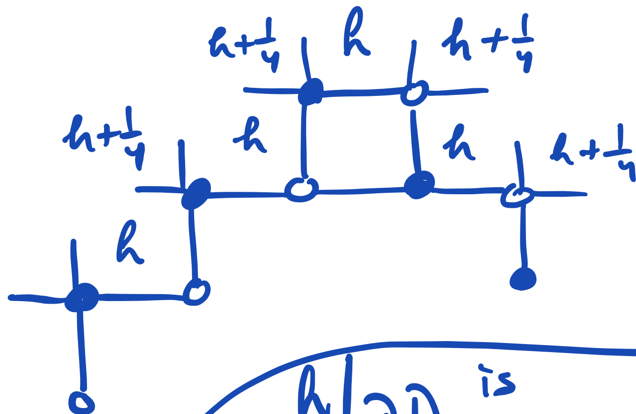
Gaussian (free Bose field)

⑦ Aztek dimond. (uniform weights)

The asymptotic of the partition function is given by the theorem.

If no dimers are sticking out, boundary value of any height function is determined by Γ and ω.





$h|_{\partial D}$ is appr. const. fixed by Γ, ω

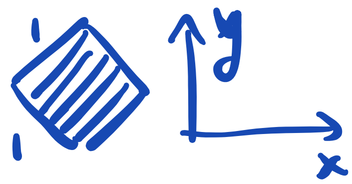
However the limit shape h_0 is not flat (Cohn, Kenyon, Propp).

Example of a domino tiling of A. diamond

$N = \frac{1}{\varepsilon}$

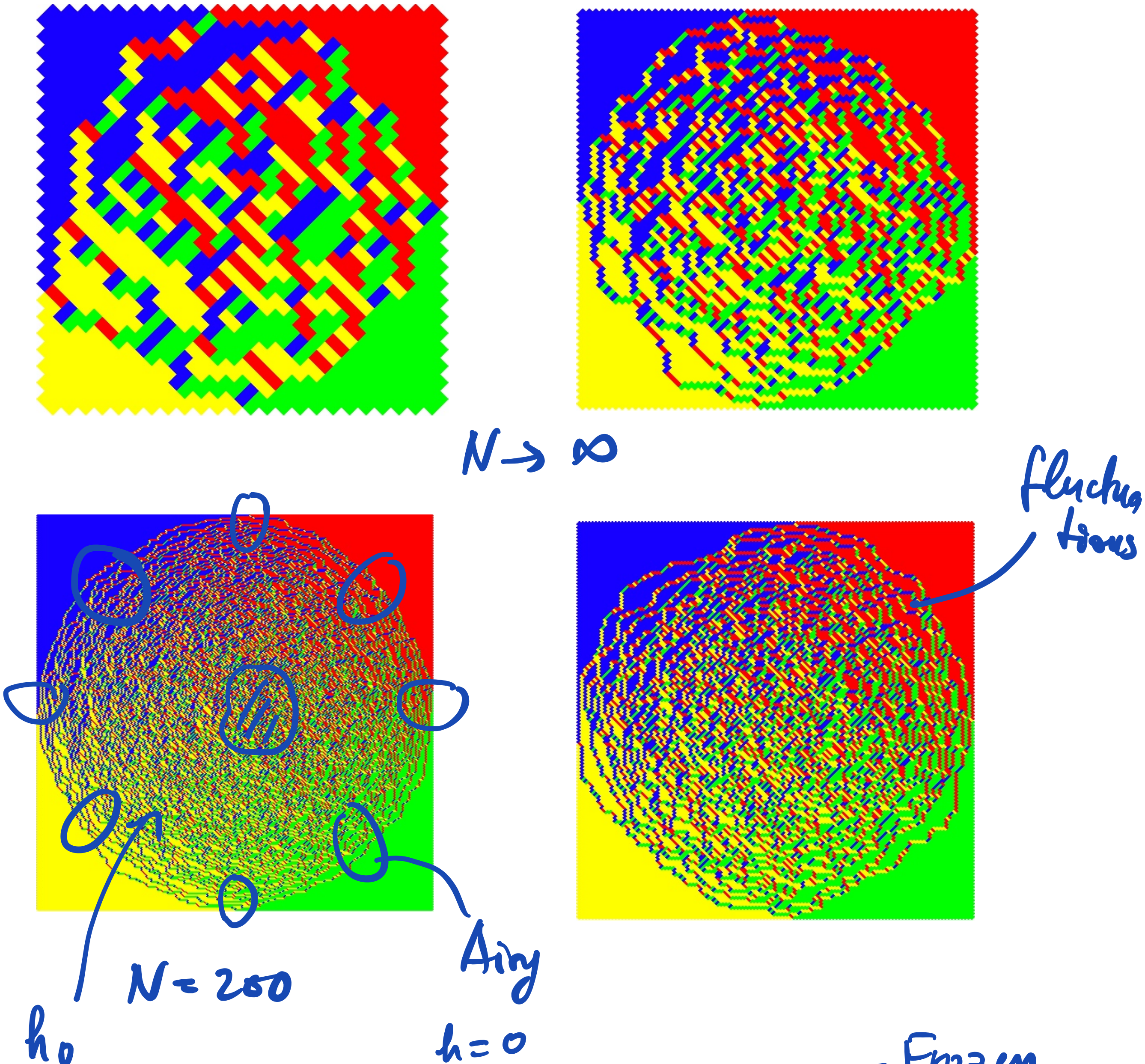
$$Z \to \exp\left(\frac{1}{\varepsilon^2} S[h]\right)$$

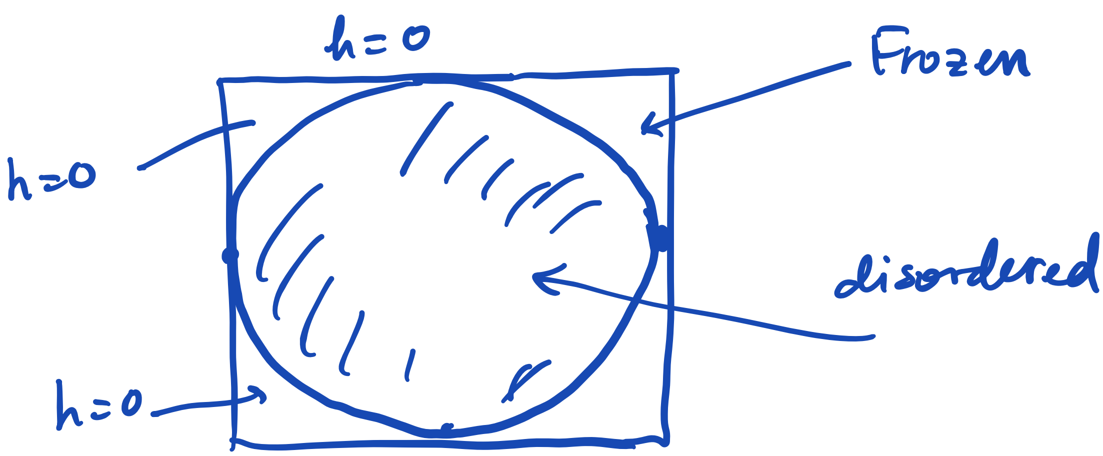
$$S[h] = \iint_D \ldots \quad \text{the same as above}$$

D


with $\varphi|_{\partial D} = 0$

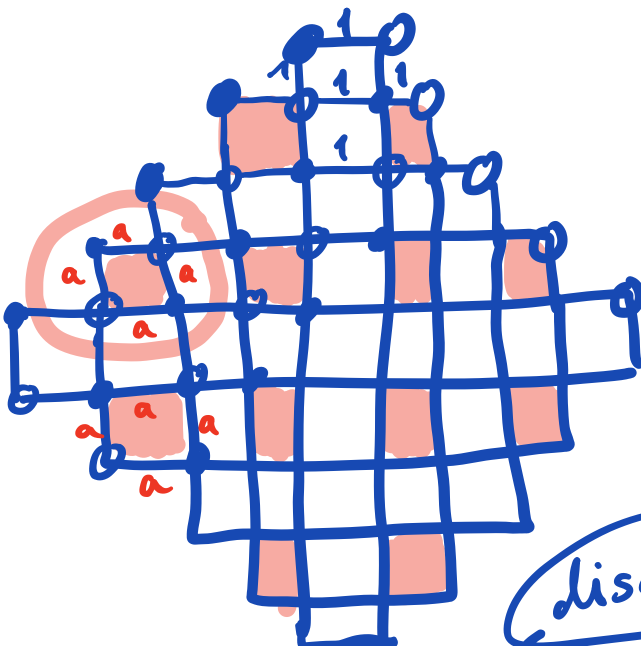
h_0 can be found explicitly





Fluctuations

8 Aztek dimond (2-periodic weights)

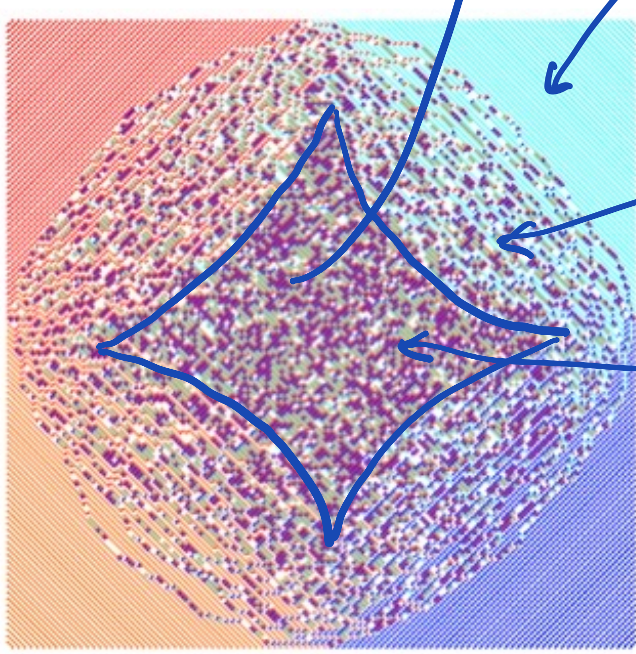


Weights:
$a > 0$
other edges have weight 1.

disappears as $a \to 1$

frozen

disordered



ordered:
limit shape is flat

Details and how to do numerical simulations are in Emily Bain's Lectures.