Lect 3

- Mapping Ising model to dimers
- Kasteleyn solution to dimer models
  (computation of Z)
- Computation of corr. fncns and
  Grassman integrals...
  exteriour alg. $\Lambda^{\bullet} V$,

Ising model: $\Gamma \subset$ Square lattice $\subset \mathbb{R}^2$
lattice domain

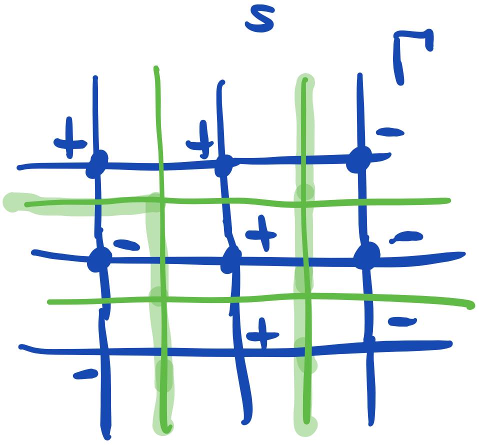
States: $s : V(\Gamma) \to \pm 1$, $v \mapsto \sigma_v$
$$\sigma_v = \pm 1$$

Energy: $E(s) = -J_1 \sum_{\substack{\text{vertical} \\ \text{edges } (v,v')}} \sigma_v \sigma_{v'} - J_2 \sum_{\substack{\text{horizontal} \\ (v,v')}} \sigma_v \sigma_{v'}$

Boltzmann probab:

$$\mathrm{Prob}(s) = \frac{e^{-\beta E(s)}}{Z}, \quad \beta = \frac{1}{kT}$$

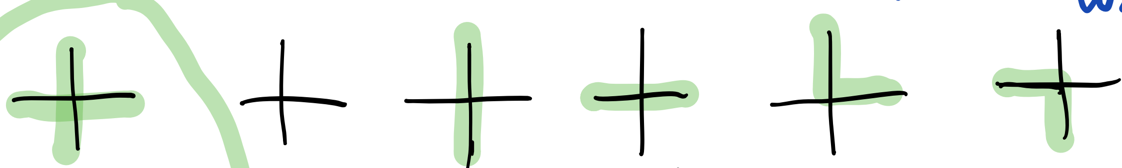
$$Z = \sum_s e^{-\beta E(s)}$$



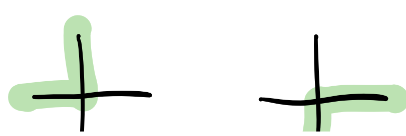
domain walls separate regions with + and regions with −

Reformulate the model in terms of domain walls.

weights of vertices in domain wall picture



$e^{2\beta J_1 + 2\beta J_2}$ $\quad 1 \quad$ $e^{2\beta J_1}$ $\quad$ $e^{2\beta J_2}$ $\quad$ $e^{\beta J_1 + \beta J_2}$ $\quad$ $e^{\beta J_1 + \beta J_2}$



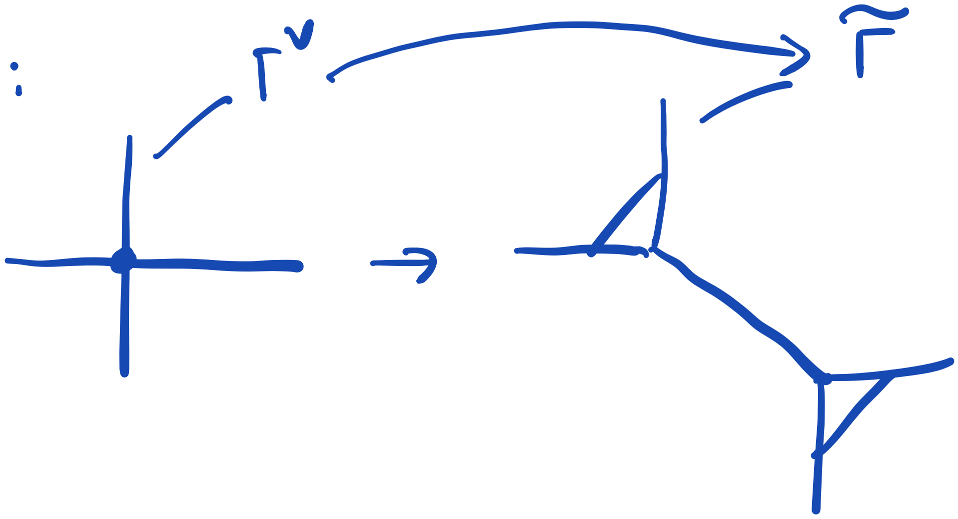
$e^{\beta J_1 + \beta J_2}$ $\quad$ $e^{\beta J_1 + \beta J_2}$

$$e^{-\beta E(s)} = \prod_{\substack{\text{vertices} \\ \text{of } \Gamma^\vee}} w\left(v, \begin{array}{c}\text{config. of domain} \\ \text{walls near } v\end{array}\right) \cdot e^{-\beta J_1 |\text{vertical edges}| - \beta J_2 |\text{horiz. edges}|}$$

Ising on Γ $\to$ Domain walls on $\Gamma^\vee$

Let us "stretch" the lattice domain

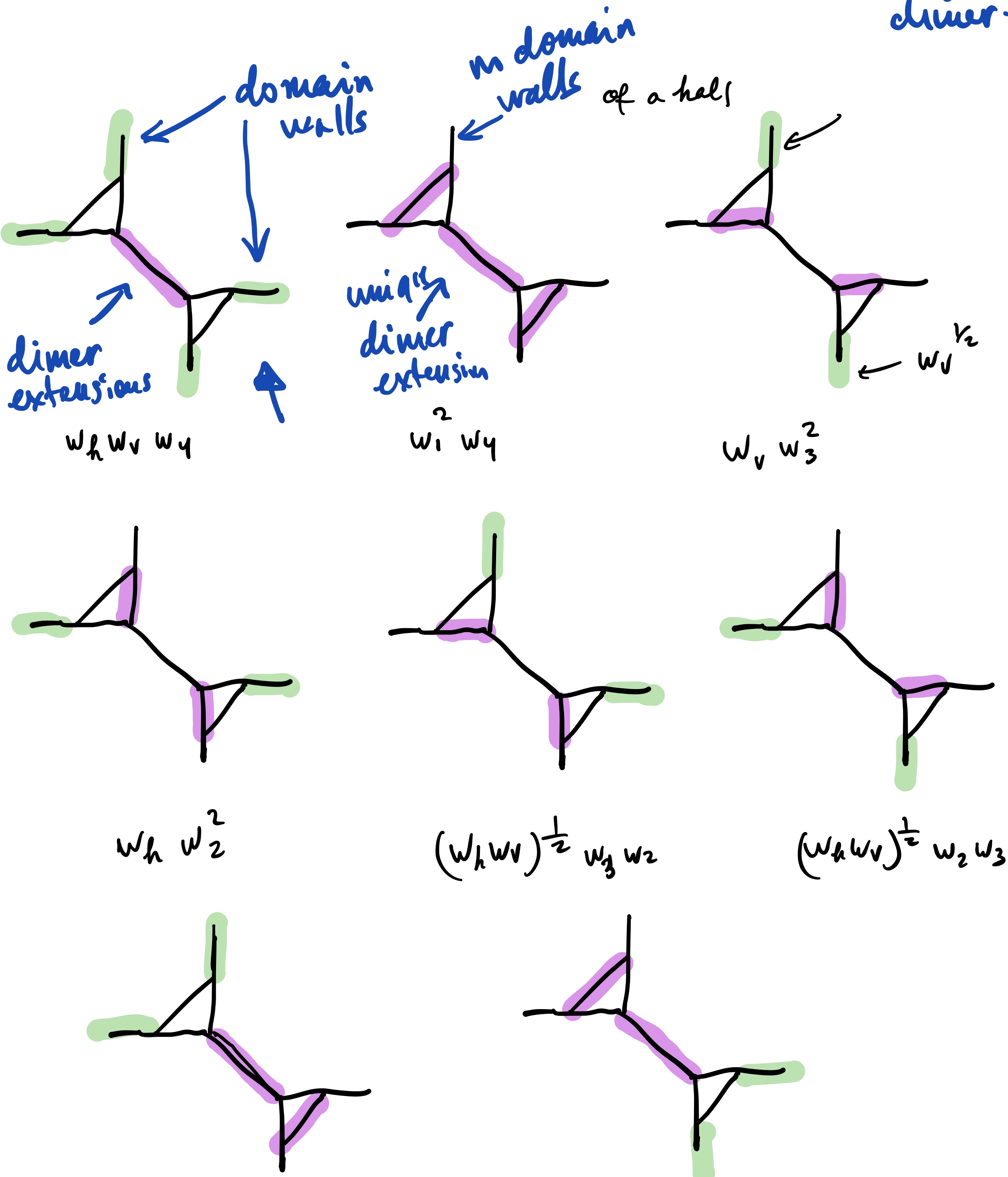
$\Gamma^\vee$:



<u>Claim</u> (dimer city)

$\exists!$ way to extend domain walls

on $\Gamma^\vee$ to <u>dimers</u> on $\tilde{\Gamma}$ (purple edges dimer)



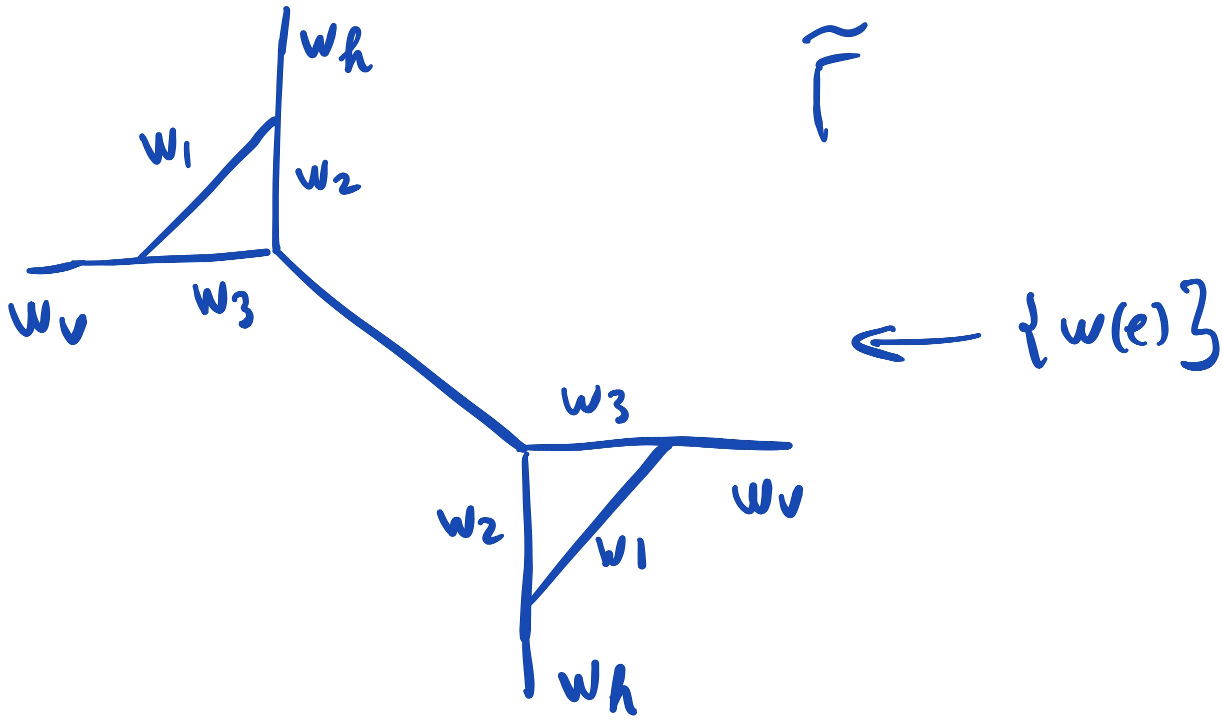
Assign weights to dimers such

$\prod w(e)$

that (weight of a dimer) =

= (weight of the corresp. domain wall config.)

see above



<u>Proposition</u>. If $w_4 = e^{2\beta(J_1+J_2)}$,

$w_3 = e^{\beta J_1}$, $w_2 = e^{\beta J_2}$,

$w_1 = e^{-\beta J_1 - \beta J_2}$, $w_v = w_h = 1$

then

(dimer weight) = (domain wall weight)

Corollary

$$Z^{\text{Ising}}_{\Gamma} = Z^{\text{domain walls}}_{\Gamma^\vee} = Z^{\text{dimers}}_{\tilde{\Gamma}}$$

More general results:

D. Cimasoni, ...

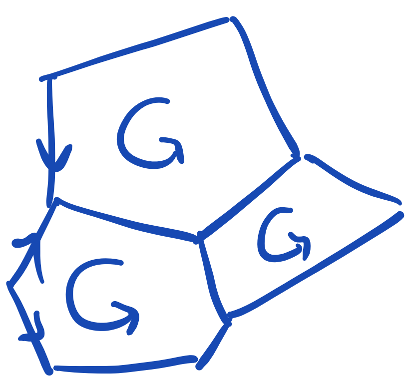
Kasteleyn solution

Compute the partition fncn in terms of the Pfaffian of a matrix.

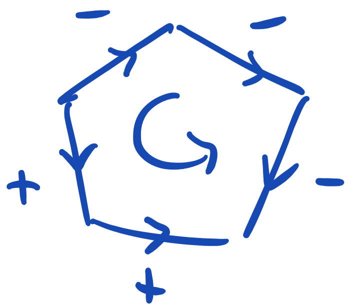
① Kasteleyn orientation of $\Gamma \subset \mathbb{R}^2$

(2-dim dimer models) (no 1-valent vertices)

$\mathbb{R}^2 \circlearrowright$



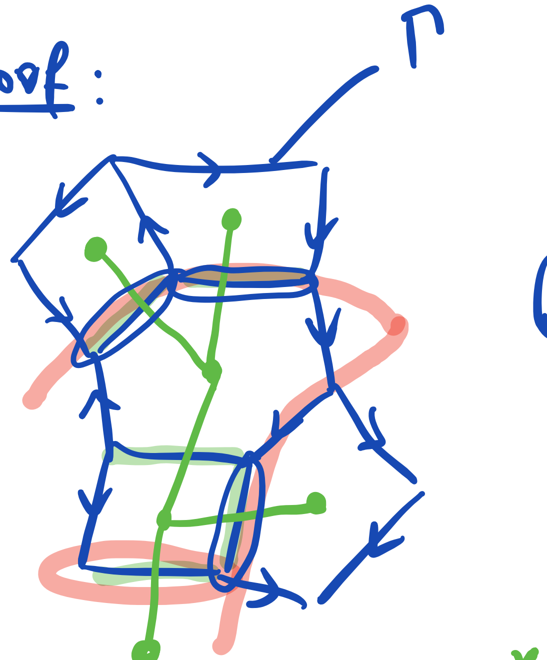
Def. An orientation of Γ is Kasteleyn if for each face f



$$\prod_{e \in \partial f} \varepsilon(e,f) = -1$$

<u>Proposition</u> Such orientations exist.

<u>Proof</u>:



choose a spanning tree T for $\Gamma^\vee$

(Thm: such T exists $T \subset \Gamma^\vee$)

<u>Step 1</u> Orient all edges of Γ which do not intersect T

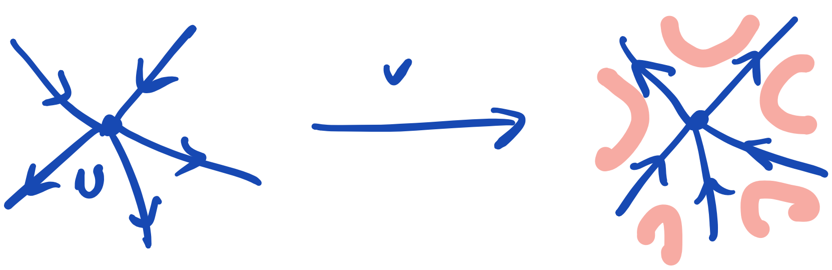
<u>Step 2</u>. Orient edges intersecting the top of T

<u>Step 3</u> Keep doing this untill all

edges of Γ are K-oriented. $\blacksquare$

<u>Corollary</u>. There are many such orientations.

<u>Def</u>. $K \sim K'$ (two Kast. orient.) if K' can be obtained from K by a sequence of orientation reversing moves



<u>Obvious:</u> if K is Kasteleyn, K_v is also Kasteleyn

$K \sim K'$ if $K' = K_{v_1 \ldots v_k}$ for some $\{v_i\}$.

$\boxed{\text{Thm. all Kasteleyn orientations of } \Gamma \subset \mathbb{R}^2 \text{ are equivalnt.}}$

($\exists!$ equivalence class of such orientations)

Proof. $\exists$ K_1 and K_2 two K-orient.

$$\varepsilon_{K_1K_2}(e) = \begin{cases} 1, & \text{if } K_1 = K_2 \text{ on } e \\ -1, & \text{otherwise} \end{cases}$$

Claim: $\prod_{e \in \partial f} \varepsilon_{K_1,K_2}(e) = 1$

h.w.

The idea: use this numbers to idenify vertices which will make the equivalence $K_1 \sim K_2$.

$\blacksquare$

Digression: If you have a

Riemannian surface $\rightsquigarrow$ spin structure

spin structure is needed to define $\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix}$

Dirac operators. (Dirac operator) $\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z} \right)$

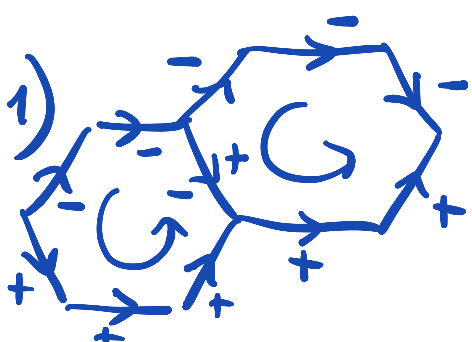
2D surfaces spin structures = complex structures

$\mathbb{R}^2$ $\exists!$ complex structure $\sim \mathbb{C}$.

The Idea: Kastelyn orient. =

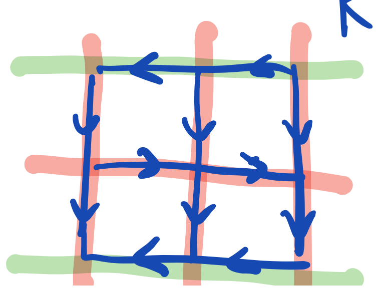
= discrete version of spin structure (of complex structures)

<u>Examples of K orientations</u>:

1) 

3 types of edges $\not\to$ $\to$ $\not\to$

K-orientation

2) 

for the square lattice

② Kastelyn matrix:

For a dimer model on Γ
we have $w : E(\Gamma) \to \mathbb{R}_{>0}$ (weights)
$e \mapsto w(e) > 0$

Define the K-matrix for a given K-orientation of Γ
$(V(\Gamma) \times V(\Gamma))$ matrix

orient

$$A^{K}_{vv'} = \begin{cases} w(v,v'), & \text{if } v \to v' \\ -w(v,v'), & \text{if } v \leftarrow v' \\ 0, & \text{if } v \text{ is not connected to } v' \end{cases}$$

Note: If we would not have $\pm$ this would be weighted adjacency matrix (analog of a Laplacian on Γ)
With $\pm$ this is a discrete version

of a 1st order operator (of a Dirac operator in the spin structure K)

<u>Thm</u> (Kasteleyn; Temperley, Fisher)

$$Pf(A^K) = \varepsilon^K \sum_{D \subset \Gamma} \prod_{e \in D} w(e), \quad \varepsilon^K = \pm 1$$

or $$Z^{\text{dimer}} = \varepsilon^K \, Pf(A^K) \quad \checkmark$$

$N \times N$ matrix
$N = |V(\Gamma)|$

<u>Reminder:</u>

A - $n \times n$ skew, n - even

$$Pf(A) = \frac{1}{2^{\frac{n}{2}} \left(\frac{n}{2}\right)!} \sum_{\sigma \in S_n} (-1)^{\sigma} \cdot$$

$$\cdot A_{\sigma_1 \sigma_2} A_{\sigma_3 \sigma_4} \cdots A_{n-1\, n}$$

Compare with $Det(A) = \sum_{\sigma \in S_n} (-1)^{\sigma} A_{\sigma_1 1} \cdots A_{\sigma_n n}$

Claim: $\quad Pf(A)^2 = Det(A).$

Proof. The idea:

$$Pf(A^K) = \underbrace{\frac{1}{2^{\frac{n}{2}}(\frac{n}{2})!}} \sum_{\sigma \in S_n} (-1)^\sigma \, \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{n-1}\sigma_n}$$

$$\underline{w(\sigma_1,\sigma_2)}\ \underline{w(\sigma_3,\sigma_4)} \cdots \underline{w(\sigma_{n-1},\sigma_n)}$$

Dimers = perfect matching on vertices
<u>connected by edges.</u>

$(\sigma_1,\sigma_2) \sim (\sigma_2,\sigma_1)$ matches σ_1 & σ_2
(dimer between σ_1 & σ_2)

$(\sigma_1,\sigma_2) \longleftrightarrow (\sigma_3,\sigma_4)$, ...

Signs in the sum are invariant
with respect to $(\sigma_1,\sigma_2) \Longleftrightarrow (\sigma_2,\sigma_1)$
$(\sigma_i,\sigma_{i+1}) \Longleftrightarrow (\sigma_j,\sigma_{j+1})$ the product of ε^K_{-}
signs does not change and $(-1)^\sigma$
also does not change

$$(-1)^{\sigma}\, \varepsilon^{K}_{\sigma_1\sigma_2} \cdots \varepsilon^{K}_{\sigma_{n-1}\sigma_n} = \varepsilon^{K}_{D}$$

depends only on the perect matching

$D \simeq$ a class in $S_n / \mathbb{Z}_2^{\frac{n}{2}} \times S_{\frac{n}{2}}$

$(\sigma_i \sigma_{i+1}) \leftrightarrow (\sigma_{i+1} \sigma_i)$

$$Pf(A^K) = \sum_{\Gamma \supset D \in S_n / \mathbb{Z}_2^{n/2} \times S_{\frac{n}{2}}} \varepsilon^{K}_{D} \prod_{e \in D} w(e)$$

perfect matching

Difficult part:

<u>Claim</u> ε^{K}_{D} does not depend on D. $\varepsilon^{K}_{D} = \varepsilon^{K}$ for all D

some ± 1

$$Pf(A)^2 = Det(A),$$

$$Z^{dimer} = \varepsilon^K \, Pf(A^K), \quad \varepsilon^K = \pm 1$$

$$\underset{\circ}{\vee}$$

$$(Z^{dimer})^2 = Pf(A^K)^2 = Det(A^K)$$

<u>Corollary</u>

$$Z^{dimer} = |Det(A^K)|^{1/2}$$

$$\underset{\circ}{\vee}$$

$$Z^{dimer} = \prod_i |\lambda_i^K|^{1/2}$$

<u>Digression on Grassman integrals.</u>

Consider $\mathbb{C}^n$ basis $\{a_i\}$ $i = 1 \dots n$,

$$\Lambda^\bullet \mathbb{C}^n = \langle a_i \mid a_i a_j = -a_j a_i \rangle$$

Choose a basis $a_1 \dots a_n$ in $\Lambda^n \mathbb{C}^n$

(secretly Schur-Weyl duality is involved here)

Any $f \in \Lambda^\cdot \mathbb{C}^n$ can be written as

$$f = f_n a_1 \cdots a_n + \text{lower order terms.}$$

Def. $$f_n = \int f \, da$$

the integral of f over the Grassman algebra $\Lambda^\cdot \mathbb{C}^n$.

Remark. $da = da_n \cdots da_1$

with rules

$$\int a_i da_i = 1, \quad \int da_i = 0, \quad a_i da_j = -da_j a_i$$

$$\int a_1 \cdots a_n \, da_n \cdots da_1 = 1$$

Lemma $A = (a_{ij})_1^n$ skew symmetric $n \times n$ matrix, then $\Lambda^\cdot \mathbb{C}^n$ (fermions)

$$\boxed{\int \exp\left(\tfrac{1}{2}(a, Aa)\right) da = \mathrm{Pf}(A)}$$

$\varepsilon \Lambda a$

Remark This is an exterior algebra version of Gaussian integral

$B > 0$

$B^t = B$,

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(x, Bx)} d^n x = \frac{C}{\sqrt{\det(B)}} \quad \text{(bosons)}$$

H.W. Prove:

- $\mathrm{Pf}(A)^2 = \mathrm{Det}(A)$

use: in $\Lambda^{\cdot}\mathbb{C}^n \otimes \Lambda^{\cdot}\mathbb{C}^n$
$\{a\} \quad \{b\}$

$$\boxed{\int e^{(a, Ab)} da\, db = (\pm)\, \mathrm{Det}(A)}$$

$\mathrm{Pf}^2(A) =$ double integral

- . . . .

Lemma. $A^t = -A$, $n \times n$, n-even

(1) $$\frac{\partial^k \mathrm{Pf}(A)}{\partial a_{i_1 j_1} \cdots \partial a_{i_k j_k}} = (-1)^{\sigma(I)} \mathrm{Pf}(A_I)$$

$\boxed{I = \{(i_1 j_1) \cdots (i_k j_k)\}}$ no coincidences $= \sigma(\overbrace{\cdots}^{\text{increasing}})$

the matrix where corresp. rows and columns removed

(2) $$(-1)^{\sigma(I)} \mathrm{Pf}(A_I) = (-1)^k \mathrm{Pf}(A) \cdot \mathrm{Pf}\left((A^{-1})_{\alpha\beta} \right)_{\alpha,\beta \in I}$$

Thm. $(i_\alpha j_\alpha) \neq (i_\beta, j_\beta)$ collection of edges in Γ (pairwise different)

$$\underline{\langle \sigma_{i_1 j_1} \cdots \sigma_{i_k j_k} \rangle} = \underline{\sum_D \mathrm{Prob}(D) \prod_{a=1}^{k} \sigma_{(i_a j_a)}(D)}$$

$$= (-1)^k a^k_{i_1 j_1} \cdots a^k_{i_k j_k} \, \mathrm{Pf}\left(\left((A^k)^{-1} \right)_{\alpha\beta} \right)$$

$\alpha, \beta \in \{i_1, \dots, i_k, j_1, \dots, j_k\}$

Proof. lemma above, see extra notes

The conclusion: For most important characteristics of a dimer model we need to find

- $\det(A^k)$
- $(A^k)^{-1}$

But this is not combinatorics

this is a linear algebra

So, we are ready to study

" $\Gamma \to \infty$ "