Lecture 1

Plan:

1. Dimer models on $D \subset \Gamma$
2. Thermodynamic limit "$D \to \infty$"
3. Limit shape phenomenon.
4. Numerical methods: how to do numeric for large systems $\left(2^{1000^2}\right)$

Lecture 1.

Equilibrium statistical mechanics

- X - finite set (set of states of a physical system)
- for each $x \in X$ we have the energy $E(x)$, $E : X \to \mathbb{R}$
- Bolzmann distribution:

T - temperature, κ - Boltzman constant

$$\mathrm{Prob}(x) \propto \exp\left(-\frac{E(x)}{\kappa T}\right)$$

Empirical law, ...

This is the main probab. "tool" "gadget" of equilibrium stat. mech.

The normalizing factor:

$$Z = \sum_{x \in X} \exp\left(-\frac{E(x)}{\kappa T}\right)$$

The partition function.

$$\mathrm{Prob}(x) = \frac{1}{Z} \exp\left(-\frac{E(x)}{\kappa T}\right)$$

$$\sum_{x \in X} \mathrm{Prob}(x) = 1$$

<u>Observables</u>: <u>functions on X</u> /

$f: X \to \mathbb{R}$, (x - random variable)

Its expectation value:

$$\mathbb{E}(f) = \sum_{x \in X} f(x)\, \mathrm{Prob}(x)$$

in physics:

$$\mathbb{E}(f) = \langle f \rangle = \sum_{x} \frac{1}{Z} e^{-\frac{E(x)}{kT}} f(x),$$

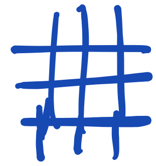
For finite X: combinatorics — — sometimes there are nice explicit formulas (rare)

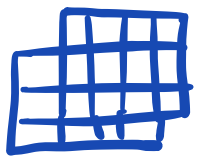
Typically: " $X \to \infty$ " interesting problems ↑

<u>Example</u>: Ising model on a 2D

lattice domain

Data:

- Lattice $\mathcal{L} \subset \mathbb{R}^2$, square: 
- Lattice domain $D \subset \mathcal{L} \subset \mathbb{R}^2$



- States are mappings s : (vertices of D) $\to \pm$, (spin up, spin down)

$\boxed{v \mapsto \sigma(v)}$

S = the space of such states (a finite set)

$$|S| = 2^{|V(D)|}$$

- Coupling constant $J \in \mathbb{R}$
- The energy function

$$E(s) = -J \sum_{\substack{\text{edges in } D \\ (v,v')}} \sigma(v)\sigma(v')$$

spins

- ferromagnetic: $J > 0$

lowest energy states:

+ + + +
+ + + +

1) $\sigma(v) = +$ $\quad E(s) = -J\,|E(D)|$

2) $\sigma(v) = -$ $\quad E(s) = -J\,|E(D)|$

\- \- \- \-
\- \- \- \-
\- \- \- \-

- antiferromagnet: $J < 0$

....

Boltzmann distribution:

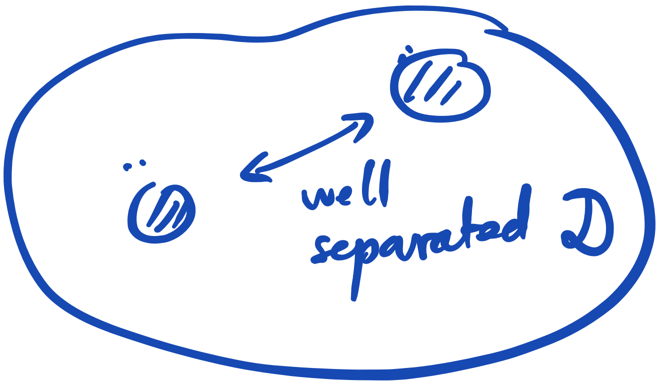
$$\mathrm{Prob}(s) = \frac{1}{Z} \exp\left(-\frac{E(s)}{kT}\right)$$

$$Z = \sum_{s \in S} \exp\left(-\frac{E(s)}{kT}\right).$$

<u>Local correlation functions</u>

$$G_{v_1 \dots v_n} = \mathbb{E}\left(\sigma(v_1) \cdots \sigma(v_n)\right) =$$

$$= \sum_{s} \mathrm{Prob}(s)\, \sigma(v_1) \cdots \sigma(v_n),$$



$\mathcal{D}$ is large

$G_{v_1 \dots v_n}$ – shows how much independent events (spin distr.) are at separated subdomains.

1) $T \to 0$,

$$\mathrm{Prob}(s) \xrightarrow[T\to 0]{} \frac{1}{2}\delta_{s,s_+} + \frac{1}{2}\delta_{s,s_-}$$

all spins are $+$; all spins are $-$

frozen

This highly correlated distribution if $+$ somewhere $\Rightarrow$ $+$ everywhere.

2) $T \to \infty$, $\exp\left(-\frac{E(x)}{kT}\right) \to 1 = e^0$

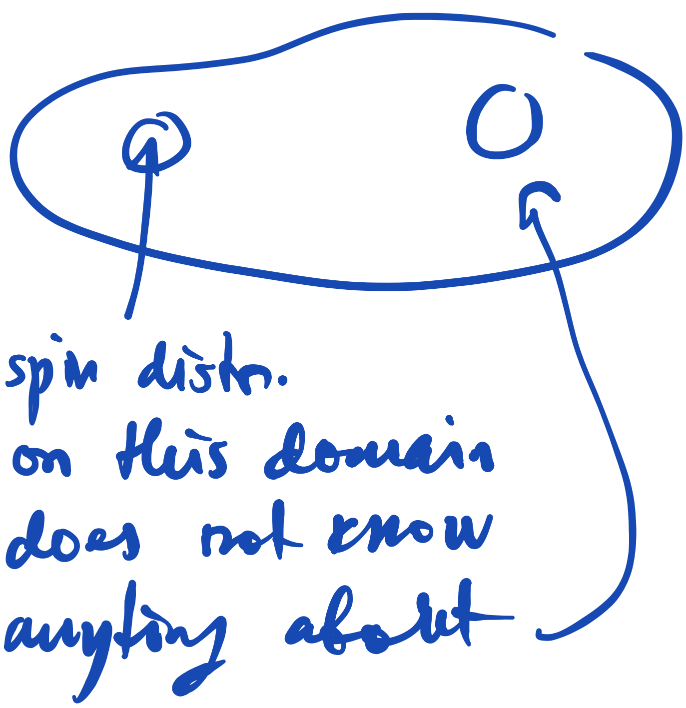
$$Z = \sum_{s \in S} e^{-\frac{E(x)}{kT}} \longrightarrow \sum_{s \in S} 1 = |S|$$

The uniform distribution of $\pm$ on vertices

$$\mathrm{Prob}(s) = \frac{1}{|S|} = \frac{1}{2^{|V(\Omega)|}}$$

The product of <u>independent</u> uniform distributions of spins on vertices.

<u>Totally decorrelated distribution</u>

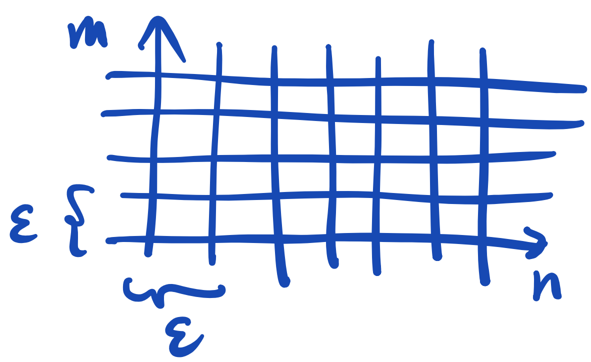


<u>Thermodynamic limit</u>

$S' = $ mappings$(V(\mathcal{D}) \longrightarrow \pm)$

"$\mathcal{D} \to \infty$" Typically this means:

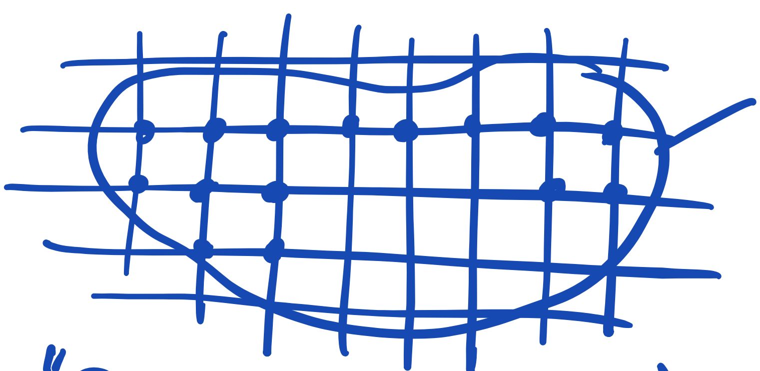
- Fix $\mathbb{D} \subset \mathbb{R}^2$ connected, simply connected domain in $\mathbb{R}^2$.
- Fix $\varepsilon > 0$, $\varphi_\varepsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$



← $\varphi_\varepsilon(\mathbb{Z}^2)$

The lattice (square) $\varphi_\varepsilon(\mathbb{Z}^2) \hookrightarrow \mathbb{R}^2$

- $\mathcal{D}_\varepsilon = \varphi_\varepsilon(\mathbb{Z}^2) \cap \mathbb{D}$ ← what fits inside $\mathbb{D}$



$\underline{\underline{\mathbb{D}}}$

- "$\mathcal{D} \to \infty$" means $\{\mathcal{D}_\varepsilon, \varepsilon \to 0\}$

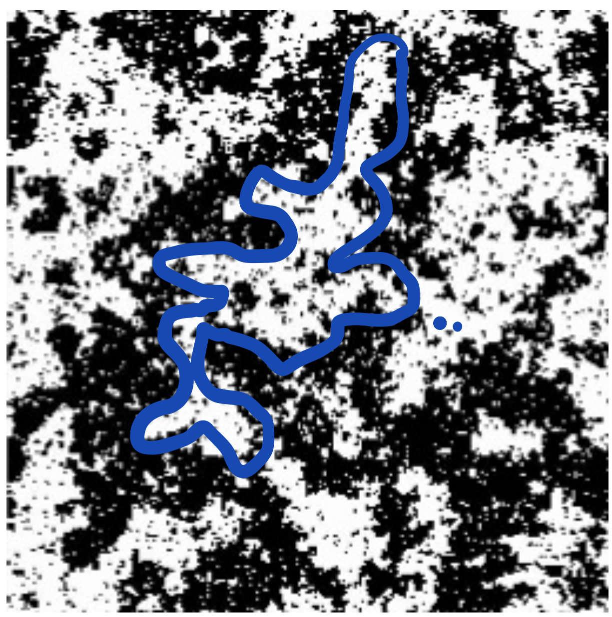
– microscopic distance = the distance on $\mathbb{Z}^2$

(discrete)

– macroscopic distance = the distance in $\mathbb{D} \subset \mathbb{R}^2$

macroscopic

$T > T_c$



Sample of a random state in the Ising model on a square

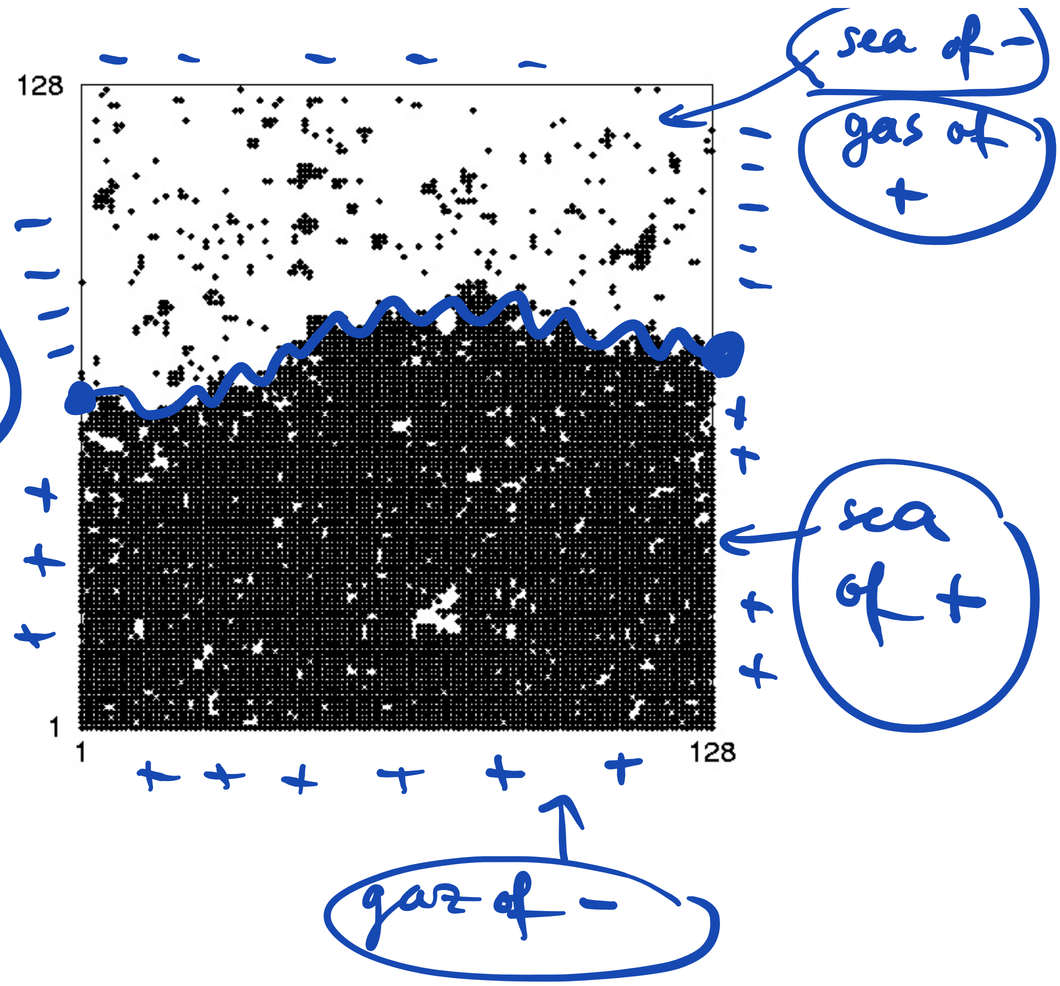
The Ising model has a phase transition

$\exists\, T_c$

Sample of a random state ______

$T < T_c$

$T > 0$



In physics:

boiling water

$T < 100C$, water

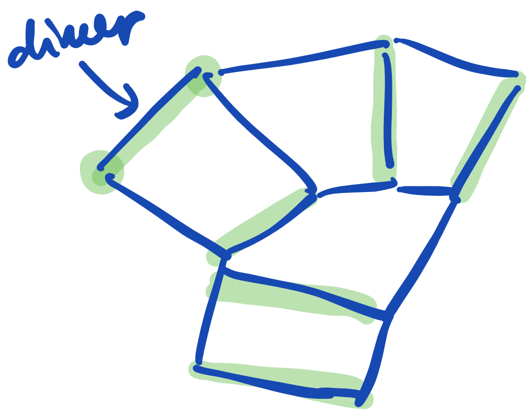
$T > 100C \to$ gas

Dimer models

Γ - finite graph

Def. A dimer configuration on Γ

is a perfect matching on vertices of Γ, connected by edges.



each vertex is connected to some other by a dimer.

perfect

- each vertex is adjacent to a dimer
- and dimers do not meet at vertices

The energy function: $E : E(\Gamma) \to \mathbb{R}$
$e \mapsto E(e)$

the energy of occupation of edge e.

The energy of a dimer state:

$$E(D) = \sum_{e \in D} E(e)$$

$$\mathrm{Prob}(D) = \frac{1}{Z}\exp\left(-\frac{E(D)}{kT}\right) =$$

$$= \frac{1}{Z}\prod_{e\in D} w(e) \qquad w(e)>0$$

$$w(e) = \exp\left(-\frac{E(e)}{kT}\right) > 0$$

$$Z = \sum_{\substack{D\subset\Gamma \\ \text{dimer config.}}} \prod_{e\in D} w(e)$$

{huge sum of Γ is large}

Local correlation functions

$$\langle \sigma_{e_1} \cdots \sigma_{e_n} \rangle = \mathrm{Prob}(e_i \subset D)$$

conditional probabilities

or (equivalently) $\left(\sigma_e : \underset{\text{space}}{(\text{dim. conf.})} \to \mathbb{R}\right)$

$$\sigma_e(D) \overset{\text{def}}{=} \begin{cases} 1, & e\in D \\ 0, & e\notin D \end{cases}$$

Claim.

1) $\langle \sigma_{e_1} \cdots \sigma_{e_n} \rangle = \mathbb{E}(\sigma_{e_1} \cdots \sigma_{e_n}) =$

$$= \sum_{\mathcal{D}} \mathrm{Prob}(\mathcal{D})\, \sigma_{e_1}(\mathcal{D}) \cdots \sigma_{e_n}(\mathcal{D})$$

2) $\langle \sigma_{e_1} \cdots \sigma_{e_n} \rangle = \prod_{i=1}^{n} w(e_i)$

$$\frac{\partial^n}{\partial w(e_1) \cdots \partial w(e_n)} \ln(Z)$$

Proof (h.w.)

The conclusion: if we know Z then we know correlation fncns.

We will do: when $\Gamma \subset \mathbb{R}^2$ then $\exists$ a formula for Z (Kasteleyn solution) in terms of $\det(|V(\Gamma)| \times |V(\Gamma)|)$.

If $\Gamma \subset \Sigma_g$, $\exists$ a version of Kastleyn formula (spin structures)

. . . . .

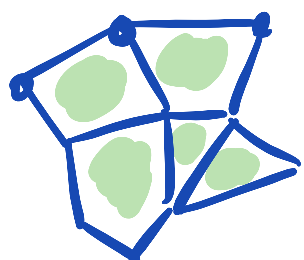
Dimers and tilings

$\exists\ \Gamma \subset \mathbb{R}^2$ plane graph, finite connected. Defines a cell complex

(alg. topology)

$X_\Gamma \equiv \Gamma$

Γ :

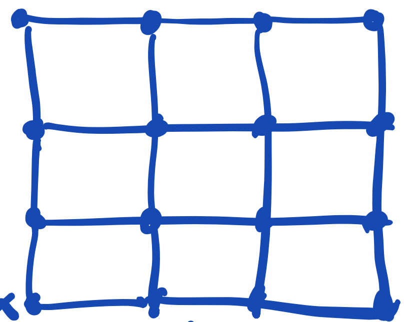


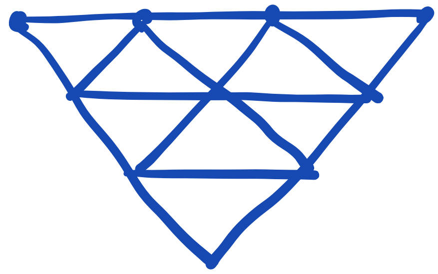
graph
- 0-cells = vertices
- 1-cells = edges
- 2-cells = "faces"

We will focus on lattice domains

$\Gamma \subset \mathcal{L} \subset \mathbb{R}^2$ (just as for Ising)

any lattice in $\mathbb{R}^2$





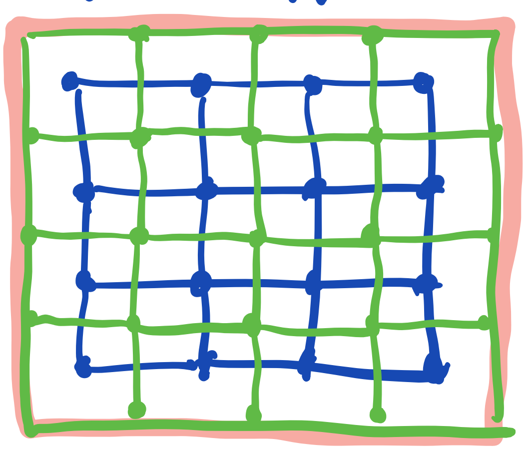
lattice domain in square lattice ← → triangular lattice

$\Gamma \subset \mathcal{L} \subset \mathbb{R}^2$

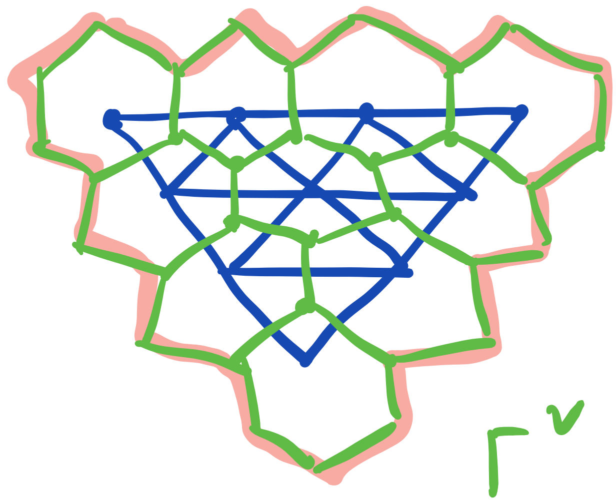
two examples of 2D cell complexes.

Dual lattice domains (<u>dual cell complexes</u>)

alg. topology



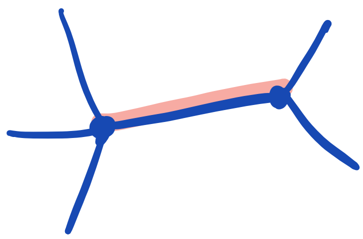
$\Gamma^\vee$



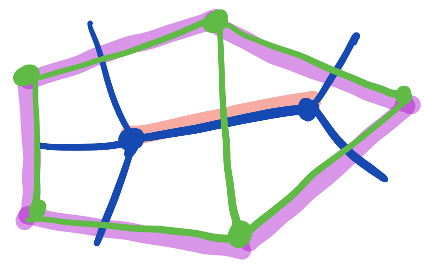
$\Gamma^\vee$

dual lattice domain to Γ

The bijection between dimers on Γ and <u>double tiles</u> on $\Gamma^\vee$



dimer on Γ



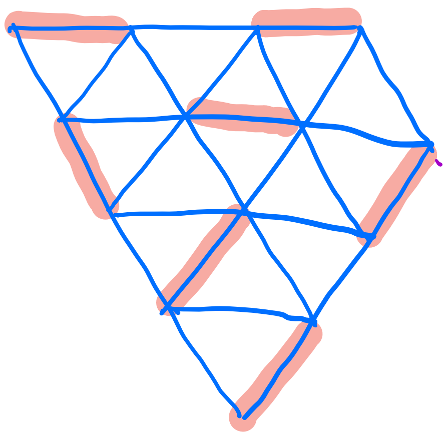
<u>double tile</u> on $\Gamma^\vee$ corresponding to a dimer on Γ

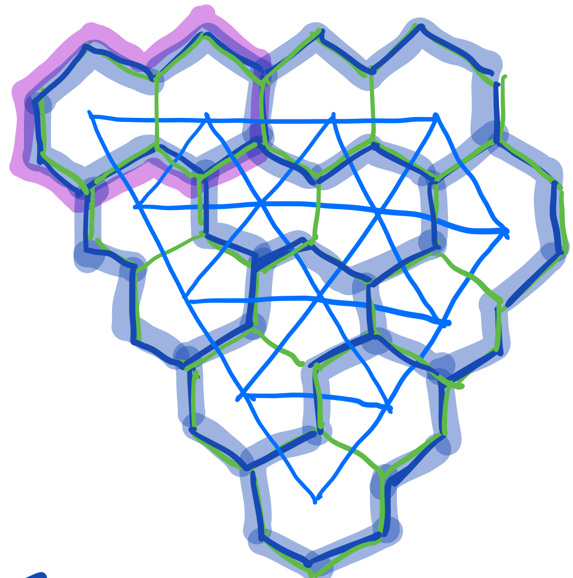
<u>double tile</u> = two adjacent faces of $\Gamma^\vee$ glued along a common edge.

<u>Theorem</u> Dimers on Γ are in bijection with double tilings of $\Gamma^\vee$.

<u>Proof</u> (h.w.)

a dimer config. on Γ





The corresponding tiling of $\Gamma^\vee$

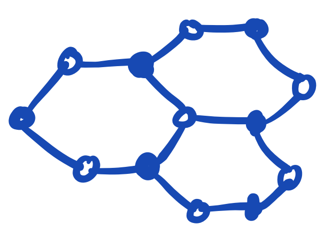
Bipartite graphs

Def. Γ is bipartite if

$$V(\Gamma) = B \sqcup W$$

s.t. black vertices are connected only to w. vertices and w. vertices are connected only to b. vertices.

Examples:



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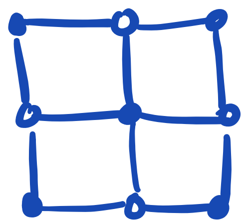
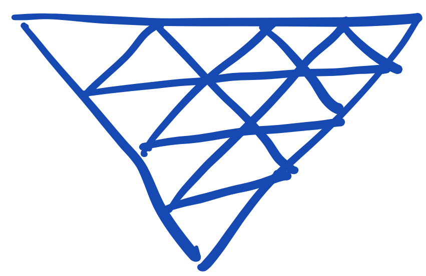
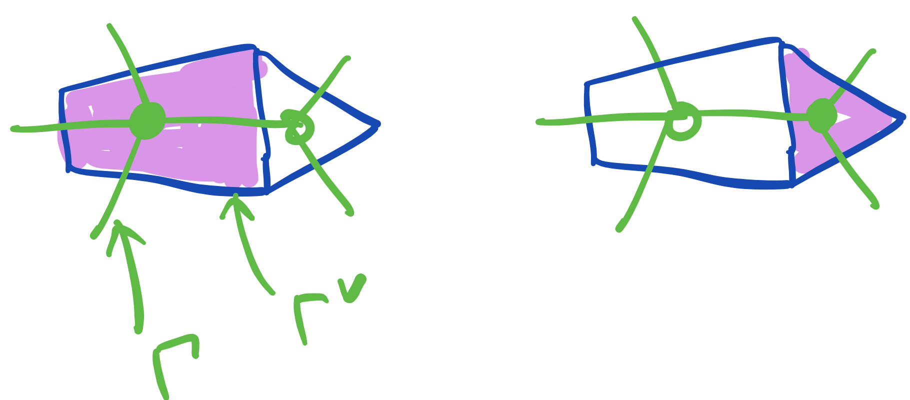


Nonexample: triangular lattice



no bipartite structure

If Γ is bipartite:

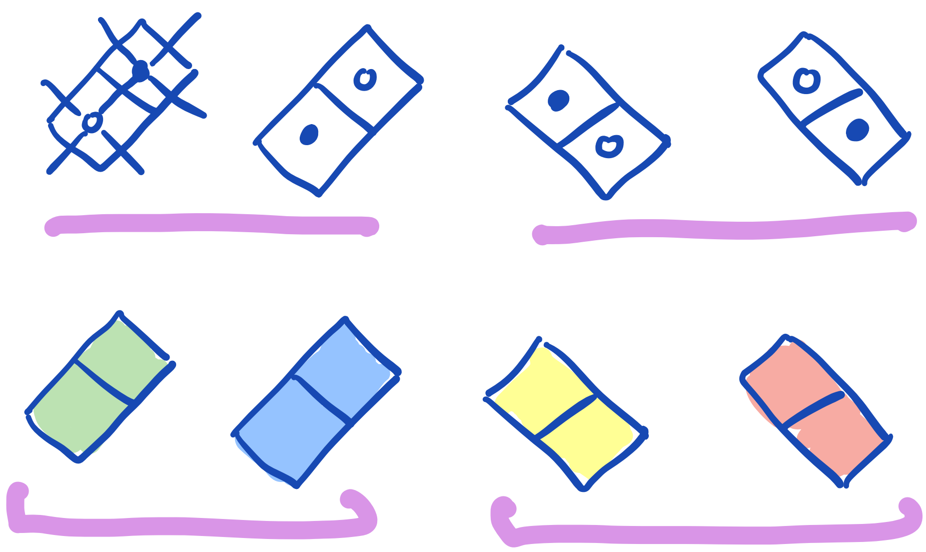


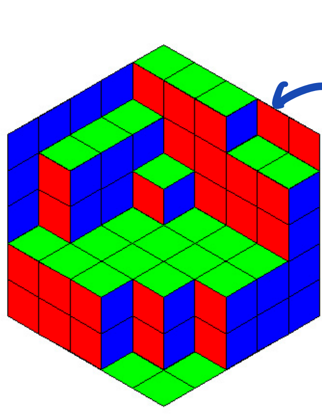
Twice more possible tiles.

Example:

dominoes on $\Gamma^\vee$

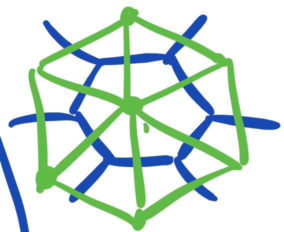
Tiling of an "Aztec diamond" by dominoes:





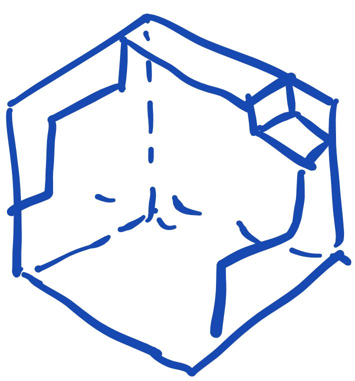
A tiling of $\Gamma^\vee$ = domain (hexagonal domain) on a triangular

Dual to a dimer config. on a lattice dual lattice domain $\Gamma \subset$ hexagonal lattice



three possible tiles

3D imagination.



pile of cubes.

The surface of this pile is the discrete surface corresponding to $\mathcal{D}$.

Dimers $\longleftrightarrow$ discrete surfaces.