Lect 2

Last time:

- Lagrangian mechanics
- Legendre transform $T_x Q \to T_x^* Q$
- $\mathcal{L}(\xi, q)$ on $TQ \xrightarrow{\text{Leg.}} H \in C^\infty(T^*Q)$
  $H(p, q)$

• 2nd order Lagr. theory:

$$\mathcal{L}(\varphi) = \frac{1}{2}(d\varphi, d\varphi) - \frac{m^2}{2}\varphi - V(\varphi)$$

Riem

φ - field. $\varphi \in \Omega^0(M)$, M - space time

Space of these fields = ∞- dim config. space.

analog of Q,

analog of $\frac{1}{2}(\dot{q}, \dot{q}) - \frac{m^2}{2}q^2 - V(q)$

$M = I$, flat time,

• Hamiltonian:

φ - scalar fields, $\pi \in \Omega^{n-1}(M)$ Riem.

$$H(\pi, \varphi) = \pi \wedge d\varphi - \frac{1}{2}\pi \wedge * \pi + \frac{m^2}{2}\varphi^2 + V(\varphi)$$

(by ∞- dim Legendre transform)

$$\text{Energy} = \int_M \underbrace{H(\pi, \varphi)}_{\in \Omega^n(M)}$$

- integrals of motion
- Poisson brackets on $\mathbb{R}^{2n}$, (p_i, q^i)

<u>Def</u>. A Poisson algebra is $(A, m, \{\cdot,\cdot\})$

A - vect. space, m - commutative alg. str. on A, $\{\cdot,\cdot\}$ - Lie algebra structure on A, s.t.

$$\{a, bc\} = \{a,b\}c + b\{a,c\}$$

<u>Example</u>: $A = C^\infty(\mathbb{R}^{2n})$, p_i, q^i

$$(fg)(x) = f(x)g(x)$$

exchange

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - (f \leftrightarrow g)\right)$$

vector field $\in \Gamma(TM)$

M - smooth manifold

bivector fields

fields

Def. $(M, p \in \Gamma(\Lambda^2 TM))$ is called a Poisson manifold if

$$\{f, g\} = \langle p, df \wedge dg \rangle$$

$f, g \in C^\infty(M)$ is a Poisson bracket on $C^\infty(M)$

For $x \in M$, $\quad p_x = \sum_{ij} p^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$

(in local coord)

basis in T_xM

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

$$\left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle = \delta_i^j,$$

its dual in T_x^*M is $\{dx^i\}$

$$\langle p_x, df(x) \wedge dg(x) \rangle = p^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

in local coord

$$\{f, g\} = p^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

It should satisfy the Jacobi:

$\{f_1\{f_2 f_3\}\} +$ cycle $= 0$

(this is a differential bilinear identity for p)

$\Omega^\bullet(M) = \Gamma(\Lambda^\bullet T^*M)$, $P(M) = \Gamma(\Lambda^\bullet TM)$

in local coordinates $\{x^i\}$, $\{\frac{\partial}{\partial x^i}\}$ - natural basis in T_xM, the dual basis in T_x^*M is $\{dx^i\}$; $\langle dx^i, \frac{\partial}{\partial x^j}\rangle = \delta^i_j$

k-forms

$$\omega_x = \sum_{i_1<\dots i_k} \omega_{i_1\dots i_k} dx^{i_1}\wedge\dots\wedge dx^{i_k},$$

k-vector fields

$$p_x = \sum_{i_1<\dots<i_k} p^{i_1\dots i_k}(x)\frac{\partial}{\partial x^{i_1}}\wedge\dots\wedge\frac{\partial}{\partial x^{i_k}}$$

$$\langle \omega_x, p_x\rangle = \sum_{i_1<\dots i_k} \omega_{i_1\dots i_k}\, p^{i_1\dots i_k},$$

<u>Example</u>: $\mathfrak{g}$ - f.d. Lie algebra,

$\mathfrak{g}^*$ - its dual vector space

on $C^\infty(\mathfrak{g}^*)$

$$\{f, g\}(\underset{\in\mathfrak{g}^*}{l}) = \langle \underset{\in\mathfrak{g}^*}{l}, [\underset{\in\mathfrak{g}}{df(l)}, \underset{\in\mathfrak{g}}{dg(l)}]\rangle$$

see HW for coord. description.

$T\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}^*$, $T_e\mathfrak{g}^* = \mathfrak{g}^* \Rightarrow$

$\Rightarrow T_e^*\mathfrak{g}^* = \mathfrak{g}$ Lie algebra

$\mathfrak{g}^*$ - Poisson manifold (Lie; Kirillov & Kostant)

Symplectic manifolds

(M, p) - Poisson manifold, locally

$$p_x = \sum_{i<j} p^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

$p: T^*M \to TM$, $(\tau, x) \mapsto (p(\tau), x)$

bundle map $\qquad \tau \in T_x^*M$, $\tau = \sum_i \tau_i dx^i$

basis

$$p(\tau) = \sum_{i<j} p^{ij}(x) \tau_i \frac{\partial}{\partial x^j} = \langle p, \tau \wedge \cdot \rangle$$

Def. (M, p) is a symplectic manifold if p is an isomorphism.

If p is an isomorphism, its inverse

$$\omega = p^{-1} : TM \to T^*M,$$

$$\omega\left(\frac{\partial}{\partial x^i}\right) = \sum_j \omega_{ij}(x)\, dx^j$$

$\sum_j \omega_{ij} p^{jk} = \delta_i^k$, the matrix $\omega(x) =$ = the inverse to the matrix $p(x)$.

$$\omega_x = \sum_{ij} \omega_{ij}(x)\, dx^i \wedge dx^j \in \Omega^2(M)$$

The symplectic form:

- $\det(\omega_{ij}) \neq 0$, nondegenerate

- **Thm.** p is Poisson iff $d\omega = 0$ (here we assume p^{-1} exists)

Proof. H.W.

Equivalent def. of sympl. manifolds:

Def. (M, ω) $\omega \in \Omega^2(M)$ is symplectic if ω is nondeg. and $d\omega = 0$

Thm. On a symplectic manifold $\exists$ an atlas $M = \bigcup_\alpha \mathcal{U}_\alpha$ with coordinate charts $\mathcal{U}_\alpha$

$$\omega\big|_{\mathcal{U}_\alpha} = \sum_i dp_{\alpha i} \wedge dq_\alpha^i, \quad (p_\alpha, q_\alpha) \text{ — local coord. on } \mathcal{U}_\alpha$$

$$\sum_i dp_\alpha \cdots =$$

Darboux coordinates

$$= \sum_i dp_\beta \cdots$$

Remark • In Darboux coordinates the Jacobian of coord. transform $(p,q) \mapsto (P,Q)$,

$$\begin{pmatrix} \frac{\partial P}{\partial p} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \end{pmatrix} \in Sp(2n)$$

- In smooth geometry $\left(\frac{\partial X}{\partial x}\right)$ invertible $\in GL_n(\mathbb{R})$
- In smooth + fix a volume form

$$\frac{\partial X}{\partial x} \in SL_n(\mathbb{R})$$

- In Riemannian $\frac{\partial X}{\partial x} \in SO(n)$

...

<u>Examples</u>: 1) $M = T^*Q_n$,

symplectic

$\{q^i\}$ - local coord. on Q_n,

$\{p_i\}$ - coord. on $T^*_q Q_n$ in the basis $\{dq^i\}$

$$\omega_{(p,q)} = \sum_i dp_i \wedge dq^i \in \Omega^2(T^*Q_n)$$

coord. nbd.

H.W. • prove ω is globally define $\omega \in \Omega^2(T^*Q_n)$

• prove that $\omega = d\alpha$ locally $\alpha = \sum_i p_i dq^i$

2) $\mathfrak{g}$ - Lie algebra, $\mathcal{O} \subset \mathfrak{g}^*$

coadjoint orbit

$Ad_g(x)$: $g e^x g^{-1} = e^{Ad_g(x)}$

$\langle Ad_g^*(\ell), x \rangle = \langle \ell, Ad_{g^{-1}}(x) \rangle$

For GL_n, $\mathfrak{gl}_n \simeq n\times n$ matrices

$$Ad_g(x) = gxg^{-1}, \qquad \mathfrak{gl}_n^* \simeq \mathfrak{gl}_n$$

$$(a,b) = tr(ab)$$

$$Ad^*_g(\ell) = g\ell g^{-1}$$

HW. prove that $\mathcal{O}$ is symplectic submanifold in $\mathfrak{g}^*$

$\underset{p,q}{\mathbb{R}^{2n}} = T^*\mathbb{R}^n$ example 1.

Hamiltonian mechanics

Hamiltonian mechanics on sympl. manifolds

(M,ω) - symplectic $\quad p = \omega^{-1}$ is Poisson

corresp. bivector field

$$\{f,g\} = \langle p, df\wedge dg\rangle$$

$\exists\, H \in C^\infty(M)$ (the Hamiltonian fncn) it generates a vector field

$(p : T^*M \to TM)$

$$v_H = p(dH)$$

$$\sum_i v_H^i(x)\frac{\partial}{\partial x^i} = p^{ij}(x)\frac{\partial H}{\partial x^j}(x)\frac{\partial}{\partial x^i},$$

v_H is called Hamiltonian vector field. if we have ω

$$\mathrm{HamVect}(M) \subset \mathrm{Vect}(M) = \Gamma(TM)$$

$\mathrm{Vect}(M)$: all vector fields

<u>Def.</u> Hamiltonian dynamical system generated by H on (M,ω) has trajectories = flow lines of v_H

$$\boxed{\frac{dx^i(t)}{dt} = v_H^i(x(t)) = p^{ij}(x(t))\frac{\partial H}{\partial x^j}(x(t))}$$

Hamilton equations

In Darbou coord. they are

old friends:

$$\begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q^i}(p,q) \\ \dot{q}^i = \dfrac{\partial H}{\partial p^i}(p,q) \end{cases}$$

This is the Hamilt. mechanics on (M, ω).

Classical observables

They are $\in C^\infty(M)$

The evolution of observables: $(F \in C^\infty(M))$

$$F_t(x) \overset{def}{=} F(x(t)),$$

$$\begin{cases} x(t) - \text{a flow line of } v_H \\ \text{s.t. } x(0) = x \end{cases}$$

Thm. $\begin{cases} \dfrac{dF_t}{dt} = \{H, F_t\} \\ F_0 = F \end{cases}$

Proof. HW (follows immediately from $\frac{dF_t(x)}{dt} = \frac{d}{dt}F(x(t)) = \frac{dx^i(t)}{dt}\frac{\partial F}{\partial x^i}(x)\Big|_{x=x(t)}$

. . . . ∎

Formal solution:

$$F_t = \sum_{n=0}^{\infty} \frac{t^n}{n!}\{H\{\cdot\cdot\{H,F\}\cdot\cdot\}$$

(one should prove conv.) (proof. HW) $\frac{dF_t}{dt} = \{H, F_t\}$

So far:

$\mathbb{R}^n\,(q) \xrightarrow{\text{Legendre}} \mathbb{R}^{2n}(p,q) \xrightarrow{\text{Poisson manifolds}}$

Lagrangian mech. (Variational principle)

Hamilt. mech traj. = = flow lines of Ham. vector fields

$\to (M,\omega)$

Hamilt. mech, traj = flow lines of v_H

→ Variational principle for Hamilt. mechanics (Hamilton - Jacobi)

Assume (M, ω), $\omega = d\alpha$, $\alpha \in \Omega^1(M)$

<u>Example</u>: $M = T^*Q_n$,

Consider the functional (HJ action) on paths in M $(\sigma : [t_1, t_2] \to M)$

(Non-param, geom. path = $\mathrm{Im}(\sigma) \subset M$) parametrized paths (not to Q_n when $M = T^*Q_n$)

$$\boxed{S[\sigma] = \int_{\mathrm{Im}(\sigma) \subset M} \alpha - \int_{t_1}^{t_2} H(x(t))\, dt}$$

$$S[\sigma] = \int_{t_1}^{t_2} \alpha_i(x(t)) \frac{dx^i}{dt} dt - \int_{t_1}^{t_2} H(x(t)) dt$$

$$\alpha = \sum_i \alpha_i \, dx^i$$

$$\delta S[\sigma] = \int_{t_1}^{t_2} \Big(\delta x^j \frac{\partial \alpha_i}{\partial x^j}(x(t))\, \dot{x}^i + \alpha_i(x(t)) \frac{d\,\delta x^i(t)}{dt}$$

$$- \delta x^i(t) \frac{\partial H}{\partial x^i}(x(t))\Big)\, dt$$

$$= \int_{t_1}^{t_2} \Big(\delta x^j \frac{\partial \alpha_i}{\partial x^j} \dot{x}^i - \dot{x}^j \frac{\partial \alpha_i}{\partial x^j}(x(t))\, \delta x^i -$$

$$- \delta x^i \frac{\partial H}{\partial x^i} \Big)\, dt + \alpha_i(x(t))\, \delta x^i(t) \Big|_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2} \Big(\Big(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i} \Big) \dot{x}^i - \frac{\partial H}{\partial x^j} \Big)\, \delta x^j\, dt$$

$$+ \alpha_i(x(t))\, \delta x^i(t) \Big|_{t_1}^{t_2}$$

(i) the bulk (Euler-Lagr.) for HJ action:

$$\Big(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i} \Big) \dot{x}^i = \frac{\partial H}{\partial x^j} ,$$

$$\omega = d\alpha = d(\alpha_i dx^i) = \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i =$$

$$= -\sum_{i<j} \left(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i} \right) dx^i \wedge dx^j$$

$$\omega(x(t)) \frac{dx(t)}{dt} = dH(x(t))$$

$$\frac{dx(t)}{dt} = P(dH)(x(t))$$

flow line of v_H

$\Rightarrow$ (EL for HJ) $\Longleftrightarrow$ equations for flow lines of v_H (Hamilton equations)

(ii) Boundary terms: $X = x(t_2)$, $x(t_1) = x$

$$\alpha_i(x(t))\,\delta x^i(t)\Big|_{t_1}^{t_2} =$$

$$= \alpha_i(X)\,\delta X^i - \alpha_i(x)\,\delta x^i =$$

$\delta X \in T_X M$, $\delta x \in T_x M$

$$= \langle \alpha(X), \delta X \rangle - \langle \alpha(x), \delta x \rangle$$

Let us assume b.c. at t_1, t_2

are independent $\iff$ $B_1, B_2 \subset M$

and we are looking for solutions to Hamilton equations s.t.

$$x(t_2) \in B_2, \quad x(t_1) \in B_1$$
$$\overset{\shortparallel}{X} \qquad\qquad \overset{\shortparallel}{x}$$

$$\Rightarrow \quad \delta X \in \underline{T_X B_2} \subset T_X M \qquad \delta x \in T_x B_1 \subset T_x M$$

$$\Rightarrow \langle \alpha(X), \delta X \rangle = \langle \iota^*_{B_2}(\alpha), \delta X \rangle_{T_X B_2}$$

$\iota^*_{B_2}(\alpha)$ is the pull-back of α to a form on B_2.

(restriction of α on M to a form on $B_2 \subset M$)

If we want

$$\langle \alpha(X), \delta X \rangle = 0, \text{ for any } X \in B_2$$
$$\langle \alpha(x), \delta x \rangle = 0, \ \forall x \in B_1$$

this equivalent to saying:

$$\imath^*_{B_2}(\alpha)=0 \quad , \quad \imath^*_{B_1}(\alpha)=0$$

Proposition. Boundary conditions $B_1, B_2 \subset M$ are variational if

$$\imath^*_{B_1}(\alpha)=0 \ , \quad \imath^*_{B_2}(\alpha)=0$$

Remark. Pull-back commutes with the DeRham differential d $\Rightarrow$

$$\imath^*_{B_1}(d\alpha)=0 \longleftrightarrow \underline{\imath^*_{B_1}(\omega)=0}$$

$$\imath^*_{B_2}(d\alpha)=0 \longleftrightarrow \underline{\imath^*_{B_2}(\omega)=0}$$

Isotropic, coisotropic, Lagrangian submanifolds and we will see that "good" variational B.C. are <u>Lagrangian submanifolds</u>