Lect 15

Summary of what we did:

- Modular tensor categories and their properties
- Invariants of 3-manifolds via surgery along framed links and invariants of links.
- TQFT as a modular functor

$$3\text{-cob}^{(st)} \longrightarrow \underline{Vect}$$

$$\Sigma_g^{st} \mapsto \mathcal{H}(\Sigma_g^{st})$$

$$M \mapsto \tau(M) \in \mathcal{H}(\partial M^{st})$$

$$\mathcal{H}(\Sigma_g^{st}) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, H^{\otimes g}), \quad \mathcal{H}(\Sigma_g^{st})^* = \mathrm{Hom}_{\mathcal{C}}(H^{\otimes g}, \mathbb{1})$$

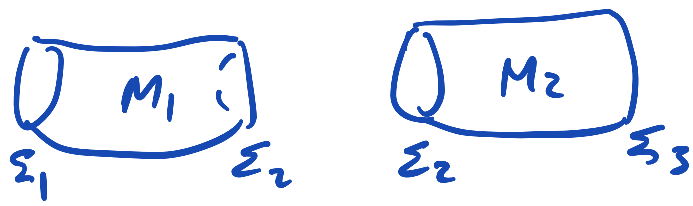
If M_L is a surgery on $S^3 \setminus \partial M^{st}$ along a framed link $L \subset S^3 \setminus \partial M^{st}$, then

$$\tau(M_L) = \text{numerical factors } \mathrm{inv}\Big(L \sqcup \big(\bigsqcup_\alpha \Gamma_\alpha\big) \subset S^3\Big)$$

where Γ_α are connected graph parametrizing connecting component $\partial_\alpha M^{st}$.

Among of what I did not do:

- Modular functor is projective



$$\tau(M_1)\,\tau(M_2) = \left(\frac{p_+}{p_-}\right)^{\lambda(M_1, M_2)} \tau(M_1 \cup_{\Sigma_2} M_2)$$

Here $\lambda(M_1, M_2)$ is the Maslov index determined by 3 Lagrangians in $H_2(W)$, W - 4-dim manifold....

- In particular, this gives a projective representation of the mapping class group of Σ_g^{st}.

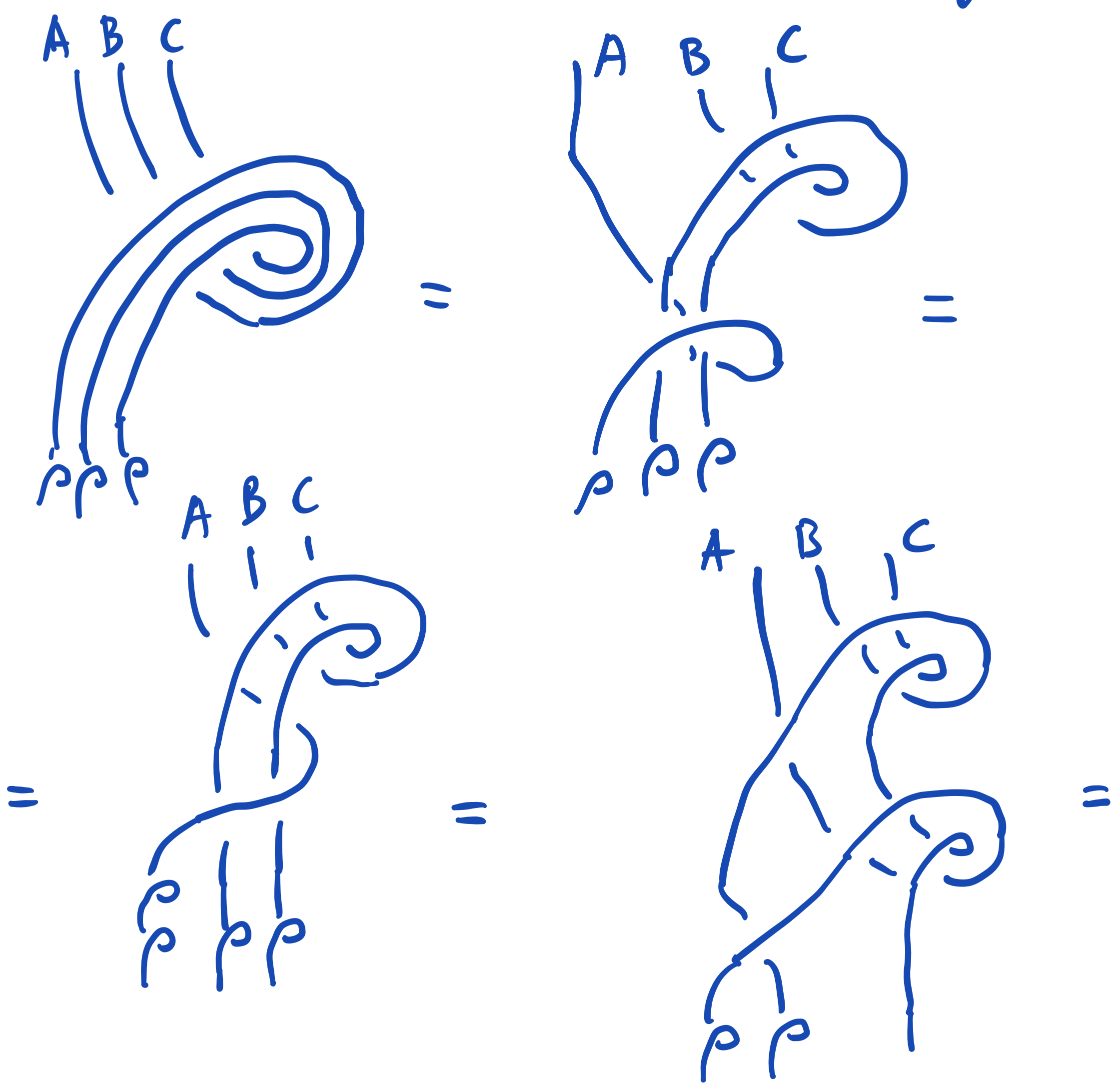
Now let us prove Vafa's theorem

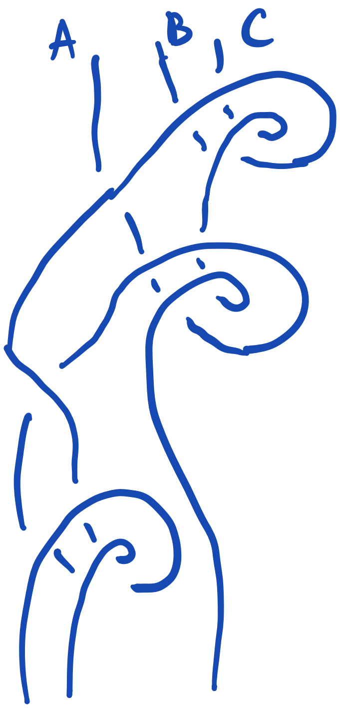
Thm $\exists\, N \in \mathbb{N}$ such that

$$v_i^N = 1 \quad \text{for each } i \in I$$

Corollary $\left(\frac{p_+}{p_-}\right)^N = 1$ for some $N \in \mathbb{N}$

Proof. First, consider the identity





$=$

or $\theta_{A\otimes B\otimes C}(\theta_A \otimes \theta_B \otimes \theta_C) =$

$$= (\mathrm{id}_A \otimes \theta_{B\otimes C})(c_{BA} \otimes \mathrm{id}_C)(\mathrm{id}_B \otimes \theta_{A\otimes C})$$

$$(c_{BA}^{-1} \otimes \mathrm{id}_C)(\theta_{A\otimes B} \otimes \mathrm{id}_C)$$

Take $A = V_i$, $B = V_j$, $C = V_k$ and let

$$N_{ijk}^{\ell} = \dim(\mathrm{Hom}_{\mathcal{C}}(V_\ell, V_i \otimes V_j \otimes V_k))$$

$$= \sum_s N_{ij}^{s} N_{sk}^{\ell} = \sum_t N_{it}^{\ell} N_{jk}^{t}$$

where $N_{ij}^{k} = \dim\left(\mathrm{Hom}_{\mathcal{C}}(V_k, V_i \otimes V_j)\right)$

Take the determinant of both sides on the space $\mathrm{Hom}_{\mathcal{C}}(V_e, V_i \otimes V_j \otimes V_k)$. Determinants of c_{BA} cancel and we have the identity on eigenvalues of v:

$$(v_i v_j v_k v_e)^{N_{ijk}^{e}} = \prod_s v_s^{N_{ijk,s}^{e}}$$

$$N_{ijk,s}^{\ell} = N_{jk}^{s} N_{is}^{\ell} + N_{js}^{e} N_{ik}^{s} + N_{ij}^{s} N_{sk}^{e}$$

write $v_j = \exp(2\pi i h_j)$ and take log of this identity. This gives an identy for h_j:

(1) $$N_{ijk}^{\ell}(h_i + h_j + h_k + h_e) = \sum_s \left(N_{ij}^{s} N_{sk}^{\ell} + N_{jk}^{s} N_{is}^{\ell} + N_{js}^{e} N_{ik}^{s}\right) h_s \mod \mathbb{Z}$$

We want to prove that these identities imply $h_k \in \mathbb{Q}$

Now we use identities $N_i N_j = \sum_k N_{ij}^k N_k = N_j N_i$ multiple timse (N_i is the matix with matrix elements $(N_i)_k^\ell = N_{ik}^\ell$).

Multiply (1) by N_ℓ and sum over ℓ. This gives

$$N_i N_j N_k (h_i + h_j + h_k) + \sum_\ell N_{ijk}^\ell N_\ell t_\ell =$$

$$= \sum_s (N_{ij}^s N_k N_s + N_{ik}^s N_j N_s + N_{jk}^s N_i N_s) h_s$$

Multiply by N_i and sum over i. After cancelling terms $(\sum_i N_i^2 h_i) N_j N_k$ on both sides of the identity we obtain

$$N N_j N_k (h_j + h_k) = N \sum_s N_{jk}^s N_s h_s$$

where $N = \sum_i N_i^2$.

Multiply this by N_j and sum over j. Cancelling terms $N \sum_j N_j^2 h_j N_k$ we are left with

$$N^2 N_K h_K = 0 \mod \mathbb{Z}$$

But both N and N_K are matrices with integer values, therefore h_K are rational numbers. $\blacksquare$

Now let us prove the Corollary. We already proved that

$$\begin{cases} p^+ T^{-1} S T^{-1} = S T S \\ p^- T S T = S T^{-1} S C \end{cases}$$

Taking determinants we have:

$$(p^+)^{|I|} = \det(T)^3 \det(S)$$

$$(p^-)^{|I|} = \det(T)^{-1} \det(S) \det(C)$$

Thus

$$\left(\frac{p^+}{p^-}\right)^{|I|} = \det(T)^4 \det(C)$$

But $\det(C)^2 = 1$ and $\det(T)$ is a

root of unity, we just proved this.

Therefore $\left(\frac{p_+}{p_-}\right)$ is a root of unity. $\blacksquare$

We did not discuss in this course many other interesting aspects in TQFT. Here are several:

1) MTC obtained from $U_q(sl_2)$-mod has:

- all $d_j > 0$ when $q = e^{\frac{i\pi}{r}}$. In this case it is expected that as $r \to \infty$ the theory can be described by a perturbative path integral for the classical topological Chern-Simons theory

- when $q = e^{\frac{i\pi m}{r}}$, some of the

d_j are negative. As $r \to \infty$
$m \neq 1$ is fixed the asymptotical
behaviour of invariants is
related to the hyperbolic volumes.
There is no good explanation
to this.

2) Turaev-Viro invariant and
triangulations of 3-manifolds.

3) Path integral & TQFT
(perturbative story)