

Lect 14

Last time:

- Formulation of 3d TQFT construction from modular tensor categories (MTC).
- Started to discuss again properties of MTC, coends.

Coends and their properties

1) A dinatural transformation α to an object $H \in \mathcal{C}$ is a collection of morphisms $\alpha_X : X \otimes X^* \rightarrow H$ such that for each $X, Y \in \mathcal{C}$ and $f : X \rightarrow Y$ the diagram

$$\begin{array}{ccc} X \otimes Y^* & \xrightarrow{\text{id} \otimes f^*} & X \otimes X^* \\ f \otimes \text{id} \downarrow & & \downarrow \alpha_X \\ Y \otimes Y^* & \xrightarrow{\alpha_Y} & H \end{array}$$

is commutative.

When $\mathcal{C}$ is semisimple choose a basis in the decomposition of X into simple objects,

$\{f_i^\alpha\}$ in $\text{Hom}(X, V_i)$, $\{g_i^\alpha\}$ in

$\text{Hom}(V_i, X)$ such that

$$g_i^\alpha \circ f_j^\beta = \delta_{ij} \delta^{\alpha\beta} \text{id}_{V_i}$$

By definition of f and g

$$\sum_{\alpha, i} f_i^\alpha \circ g_i^\alpha = \text{id}_X$$

Then define

$$\alpha_X = \bigoplus_i \left(\bigoplus_\alpha ((g_i^\alpha)^* \otimes f_i^\alpha) \right) : X \otimes X^* \rightarrow H$$

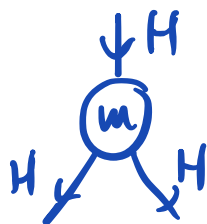
2) The algebra structure on H in $\mathcal{C}$

is a morphism $m: H \otimes H \rightarrow H$

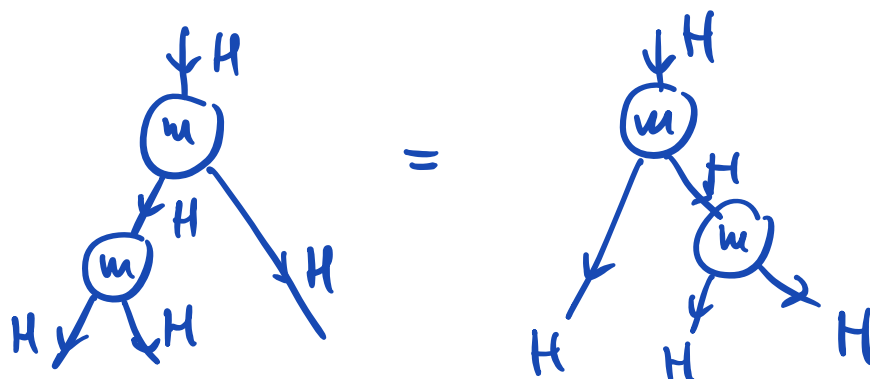
which satisfies the associativity.

Graphically it means we have a vertex

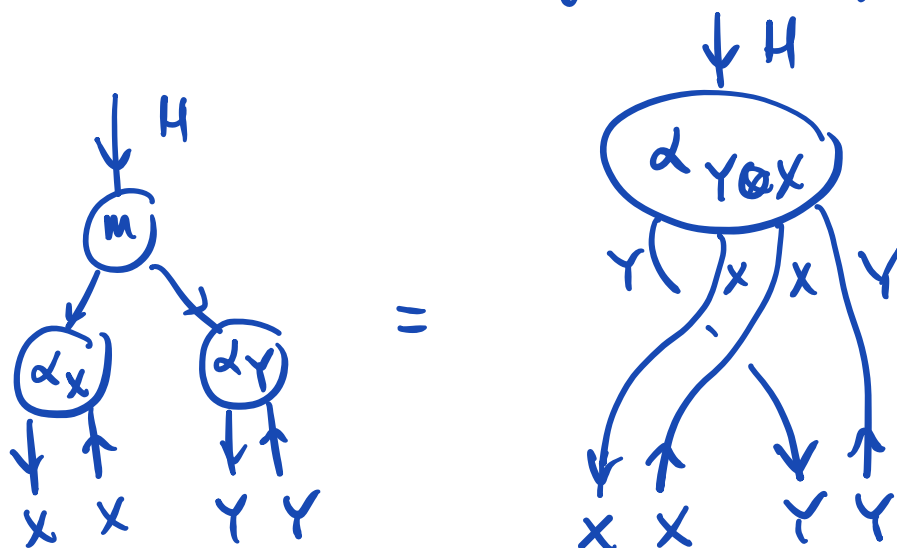
colored by m with adjacent edges
colored by H



and



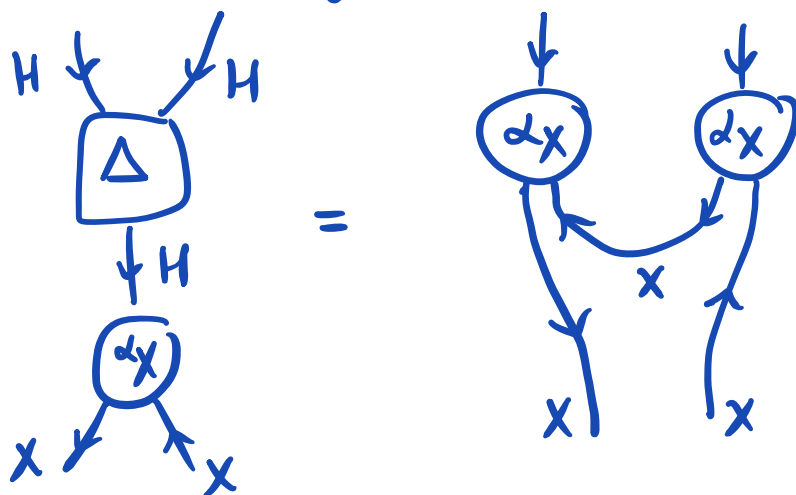
If α is a dinatural transformation
to H , define m by the diagram:



H.W. check that $\alpha_1 : 1 \rightarrow H$ is a unit in H with the multiplication defined above. Graphically it can be written as



3) The comultiplication is given diagrammatically as

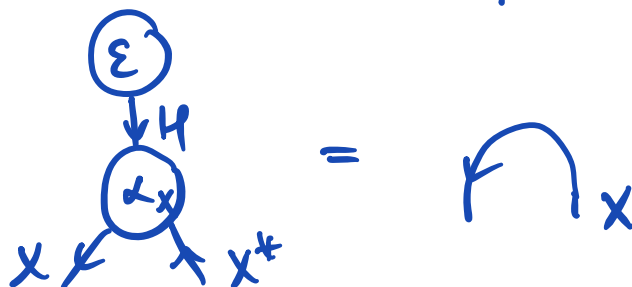


The counite is $\varepsilon : H \rightarrow 1$ defined by

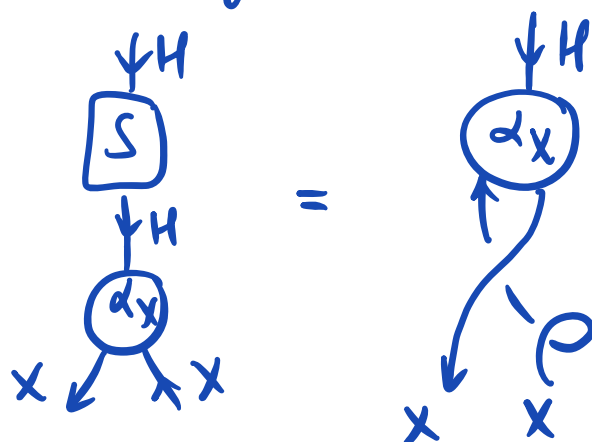
$$X \otimes X^* \xrightarrow{\alpha_X} H \xrightarrow{\varepsilon} 1$$

$\tilde{\varepsilon}_X$

Graphically: $\begin{array}{c} \textcircled{\varepsilon} \\ \downarrow H \end{array}$ defined by



The antipode is $S: H \rightarrow H$
defined by



Coends and ends exist in nonsemisimple categories too. But we will not discuss it here. The standard notation is

$$H = \int^{x \in \mathcal{C}} x \otimes x^*$$

Basic properties of MTC's . II

Let $\mathcal{C}$ be a modular tensor category ,
 i.e. $S_{ij} = \text{inv} \left(\begin{smallmatrix} i \\ \curvearrowright \end{smallmatrix} i \right)$ is nondegenerate
 and $\sum_i d_i^2 \neq 0$.

We already proved:

$$1) \quad N_{ij}^k \stackrel{\text{def}}{=} \dim_{\mathbb{C}} (\text{Hom}_{\mathcal{C}} (V_k, V_i \otimes V_j))$$

$$N_{ij}^k = N_{ji}^k = N_{i^*k^*}^{j^*} = N_{i^*j^*}^{k^*}, \quad V_{i^*} = (V_i)^*$$

$\uparrow$ braiding $\uparrow$ partial dualization $\uparrow$ total dualization

Note that N_{ij}^k are structural constants of the Grothendieck ring $[\mathcal{C}]$:

$$[V_i][V_j] = \sum_k N_{ij}^k [V_k]$$

$$2) \quad S_{i^*j^*} = S_{ij}, \quad S_{ij} = S_{ji}$$

$$3) \quad d_i d_j = \sum_k N_{ij}^k d_k$$

The mapping $[v_i] \mapsto d_i$ can be regarded as a representation of $[C]$

$$4) \quad S_{ij} = \sum_k N_{ij}^k d_k v_k v_i^{-1} v_j^{-1}$$

$$5) \quad \sum_i v_i^{\pm 1} \text{inv} \left(i \begin{array}{c} \downarrow \\ \uparrow \\ j \end{array} \right) = \rho^{\pm} v_j^{\mp 1} \downarrow j$$

$$\text{or} \quad \sum_i v_i^{\mp 1} S_{ij} \frac{1}{d_j} = \rho^{\pm} v_j^{\mp 1}$$

Define $|I| \times |I|$ matrices N_i, C

$$(N_i)_k^j = N_{ik}^j, \quad C_j^i = \delta_{i, j^*} \text{ and } X_i \in \mathbb{C}^{|I|}$$

$$(X_i)_j = \frac{1}{d_i} S_{ij} \text{ (assuming } d_i \neq 0, i \in I)$$

Note that since $*$: $I \rightarrow I$ is an involution,

$$C^2 = \text{Id}.$$

Proposition The following identities hold:

$$1) \quad N_i N_j = \sum_k N_{ij}^k N_k$$

$$2) \quad N_i^t = C N_i C$$

$$3) N_i \chi_j = S_{ij} \chi_j$$

$$4) C \chi_i = \chi_i^*$$

$$5) d_i = S_{0i}$$

Proof. 1) $\sum_k N_{ik}^e N_{jm}^k = \sum_k \dim(\text{Hom}_e(V_e, V_i \otimes V_k)$

$$\otimes \text{Hom}_e(V_k, V_j \otimes V_m)) = \dim(\text{Hom}_e(V_e, V_i \otimes V_j \otimes V_m))$$

decomposing $V_j \otimes V_m$ into the direct sum of simple objects we obtain the desired identity

2) We already proved that $N_{ik}^i = N_{ij}^{k*}$

since $\text{Hom}(V_j, V_i \otimes V_k) \simeq \text{Hom}(V_k^*, V_i \otimes V_j^*)$.

3)

$$(N_i \chi_j)_e = \sum_k N_{ie}^k \frac{S_{jk}}{d_j} = \sum_k \frac{1}{d_j} \quad \begin{array}{c} i \quad e \\ \begin{array}{c} \text{a} \\ \downarrow \\ \text{rk} \\ \uparrow \\ \text{a} \end{array} \\ j \end{array} =$$

$$\begin{aligned}
 &= \frac{1}{d_j} \text{ (diagram: a loop with nodes } i \text{ and } j \text{, with an external edge } e \text{ at node } j) = \frac{1}{d_j} \text{ (diagram: two loops sharing node } j \text{, with external edges } i \text{ and } e \text{)} = \frac{S_{ij}}{d_j} \text{ (diagram: a loop with nodes } j \text{ and } e \text{)} \\
 &= \frac{S_{ij}}{d_j} S_{je} = S_{ij} (X_j)_e
 \end{aligned}$$

$$4) (C X_i)_j = \frac{1}{d_i} S_{ij}^* = \frac{1}{d_i^*} S_{i^*j} = (X_{i^*})_j$$

$$5) d_i = \bigcirc^i = S_{0i} \text{ by def. of } S_{ij}$$

Proposition The matrices S_{ij} , $T_{ij} = \delta_{ij} \mathcal{V}_i$,

$C_{ij}^* = \delta_{ij}^*$ satisfy relations

$$p^+ T^{-1} S T^{-1} = S T S$$

$$p^- T S T = S T^{-1} S C$$

Proof. We already know that

$$\text{(diagram: a loop with a node labeled } \pm 1 \text{ and an external edge } i) = p^\pm \text{(diagram: a loop with a node labeled } \pm 1 \text{ and an external edge } i), \text{ where } \text{---} = \sum_i d_i \rightarrow^i$$

is the Kirby color and

$$\begin{array}{c} \downarrow \\ \textcircled{+1} \\ \downarrow \end{array} = \downarrow \downarrow \downarrow, \quad \begin{array}{c} \downarrow \\ \textcircled{-1} \\ \downarrow \end{array} = \downarrow \downarrow \downarrow \text{ are twists of}$$

framing. From here follows that

$$\begin{array}{c} \downarrow i_1 \quad \downarrow i_n \\ \textcircled{\pm 1} \\ \downarrow \quad \downarrow \end{array} = p^{\pm} \begin{array}{c} \downarrow i_1 \quad \downarrow i_n \\ \textcircled{\mp 1} \\ \downarrow \quad \downarrow \end{array}$$

where

$$\begin{array}{c} \downarrow \dots \downarrow \\ \textcircled{+1} \\ \downarrow \dots \downarrow \end{array} = \downarrow \dots \downarrow, \quad \begin{array}{c} \downarrow \dots \downarrow \\ \textcircled{-1} \\ \downarrow \dots \downarrow \end{array} = \downarrow \dots \downarrow$$

As a consequence, we have:

$$\begin{aligned} (*) \quad \begin{array}{c} \downarrow i \quad \downarrow k \\ \textcircled{+1} \\ \downarrow \end{array} &= p^+ \begin{array}{c} \downarrow i \quad \downarrow k \\ \textcircled{-1} \\ \downarrow \end{array} \\ &= p^+ \begin{array}{c} \downarrow i \quad \downarrow k \\ \textcircled{-1} \\ \downarrow \end{array} = p^+ \tilde{\nu}_i^{-1} \nu_k^{-1} \begin{array}{c} \downarrow i \quad \downarrow k \\ \textcircled{-1} \\ \downarrow \end{array} = \end{aligned}$$

$$= p^+ v_i^{-1} S_{ik} v_k^{-1} d_i^{-1} \downarrow^i = \frac{p^+}{d_i} (T^{-1} S T^{-1})_{ik} \downarrow^i$$

Now, evaluate (*) by definition of Kirby color:

$$\begin{aligned} & \text{Diagram with a circle, a vertical line labeled } i, \text{ a loop labeled } k, \text{ and a twist labeled } +1 \text{ on the line } i. \\ &= \sum_j d_j v_j \text{ Diagram with a circle, a vertical line labeled } i, \text{ a loop labeled } k, \text{ and a twist labeled } j \text{ on the line } i. \\ &= \sum_j d_j v_j \frac{S_{jk}}{d_j} \text{ Diagram with a circle, a vertical line labeled } i, \text{ a loop labeled } k, \text{ and a twist labeled } j \text{ on the line } i. \\ &= \sum_j v_j S_{jk} S_{ij} \frac{1}{d_i} \downarrow^i \\ &= (STS)_{ik} \frac{1}{d_i} \downarrow^i \end{aligned}$$

Thus, $STS = p^+ T^{-1} S T^{-1}$.

Similarly, changing +1 twist to -1 we obtain the second identity.

Corollary $S^2 = p^+ p^- C$

Proof. From the proposition above we have:

$$p^+ \bar{T}^1 S = S T S T$$

$$p^- S T S T = S^2 \bar{T}^1 S C$$

Thus, $p^+ p^- \bar{T}^1 S = S^2 \bar{T}^1 S C$

but $SC = CS$, $TC = CT$, therefore

$$p^+ p^- = S^2 C$$

Now use $C^2 = \text{Id}$.



Proposition The following hold:

(i) $\textcircled{\downarrow^i} = p^+ p^- \delta_{i,0} \downarrow^i$

(ii) $p^+ p^- = \sum_i d_i^2$

(iii) $\textcircled{\downarrow^i \uparrow^j} = \delta_{ij} \frac{p^+ p^-}{d_i} \begin{matrix} \curvearrowright^i \\ \curvearrowleft_i \end{matrix}$

Proof. (i)

let us close the i -strand

$$\sum_k d_k \text{ (diagram: a circle with an incoming strand labeled } i \text{ and an outgoing strand labeled } k \text{)} = \sum_k d_k S_{ki} = \sum_k S_{0k} S_{ki}$$

$$= (S^2)_{oi} = p^+ p^- (C)_{oi} = p^+ p^- \delta_{i,0}$$

(ii) let $i=0$ in (i), $p^+ p^- = \sum_k \text{ (diagram: a circle with an incoming strand labeled } k \text{ and an outgoing strand labeled } 0 \text{)} d_k =$

$$= \sum_k d_k^2$$

(iii) $\text{ (diagram: a circle with two incoming strands labeled } i \text{ and } j \text{, and two outgoing strands labeled } i \text{ and } j \text{)} = \sum_{k,a} \text{ (diagram: a circle with an incoming strand labeled } i \text{ and an outgoing strand labeled } j \text{, and a vertex labeled } a \text{)} = p^+ p^- \sum_a \text{ (diagram: a vertex labeled } a \text{ with two incoming strands labeled } i \text{ and } j \text{, and two outgoing strands labeled } i \text{ and } j \text{)} =$

$$= p^+ p^- \frac{1}{d_i} \text{ (diagram: a circle with two incoming strands labeled } i \text{ and } j \text{, and two outgoing strands labeled } i \text{ and } j \text{)} \delta_{ij},$$

orthonormal projector
 $V_i \otimes V_j \rightarrow \mathbb{1}$