Lect 14

Last time:

- Formulation of 3d TQFT construction from modular tensor categories (MTC).
- Started to discuss again properties of MTC, coends.

Coends and their properties

1) A *dinatural transformation* α to an object $H \in \mathcal{C}$ is a collection of morphisms $\alpha_X : X \otimes X^* \to H$ such that for each $X, Y \in \mathcal{C}$ and $f : X \to Y$ the diagram

$$\begin{tikzcd}
X \otimes Y^* \arrow[r, "id \otimes f^*"] \arrow[d, "f \otimes id"'] & X \otimes X^* \arrow[d, "\alpha_X"] \\
Y \otimes Y^* \arrow[r, "\alpha_Y"] & H
\end{tikzcd}$$

is commutative.

When $\mathcal{C}$ is semisimple choose a basis in the decomposition of X into simple objects, $\{f_i^\alpha\}$ in $\mathrm{Hom}(X, V_i)$, $\{g_i^\alpha\}$ in $\mathrm{Hom}(V_i, X)$ such that

$$g_i^\alpha \circ f_j^\beta = \delta_{ij}\,\delta^{\alpha\beta}\, id_{V_i}$$

By definition of f and g

$$\sum_{\alpha, i} f_i^\alpha \circ g_i^\alpha = id_X$$

Then define

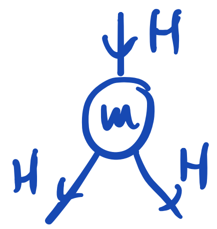
$$\alpha_X = \bigoplus_i \left(\bigoplus_\alpha \left((g_i^\alpha)^* \otimes f_i^\alpha \right) \right) : X \otimes X^* \to H$$

2) The algebra structure on H in $\mathcal{C}$ is a morphism $m : H \otimes H \to H$

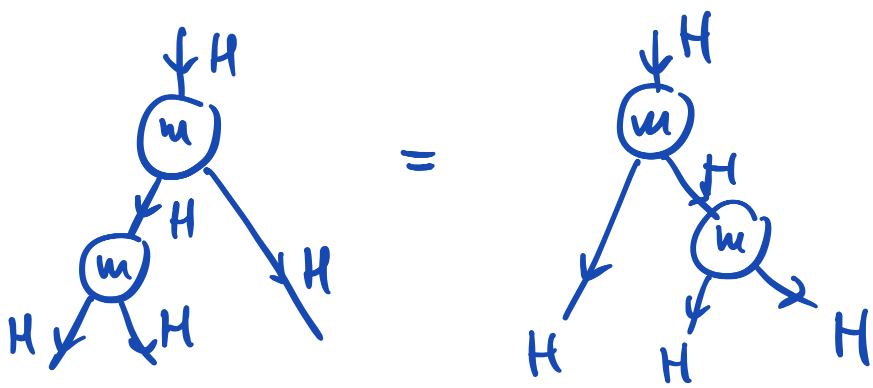
which satisfies the associativity.

Graphically it means we have a vertex

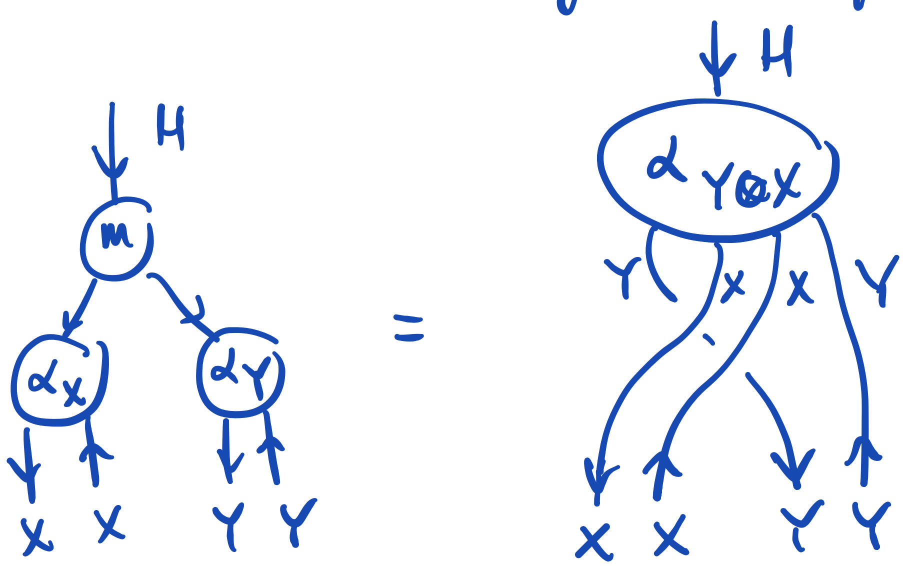
colored by m with adjacent edges colored by H



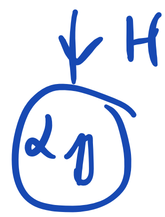
and



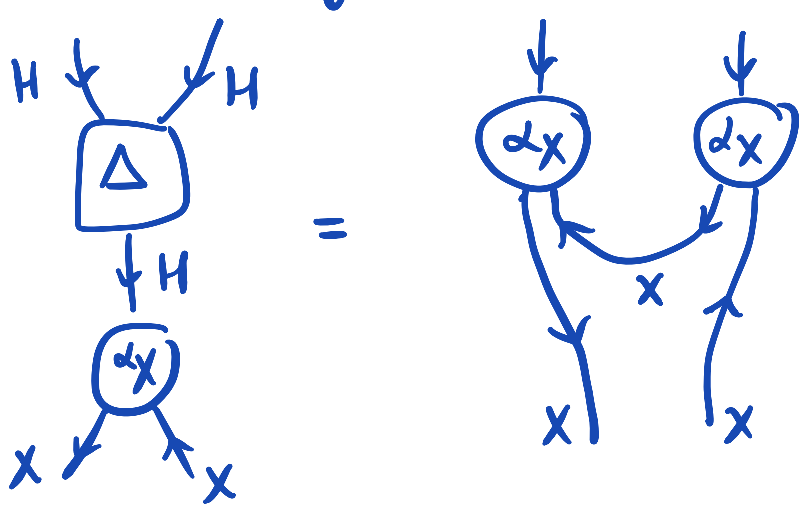
If α is a dinatural transformation to H, define m by the diagram:



H.W. check that $\alpha_{\mathbb{1}} : \mathbb{1} \to H$ is a unit in H with the multiplication defined above. Graphically it can be written as



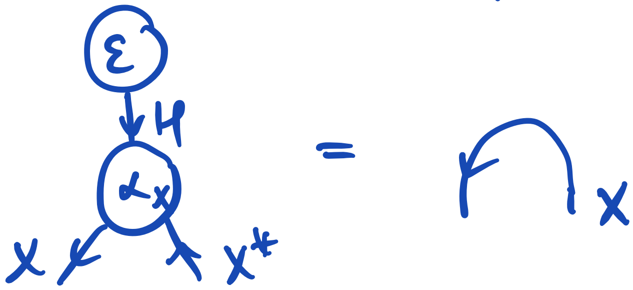
3) <u>The comultiplication</u> is given diagrammatically as



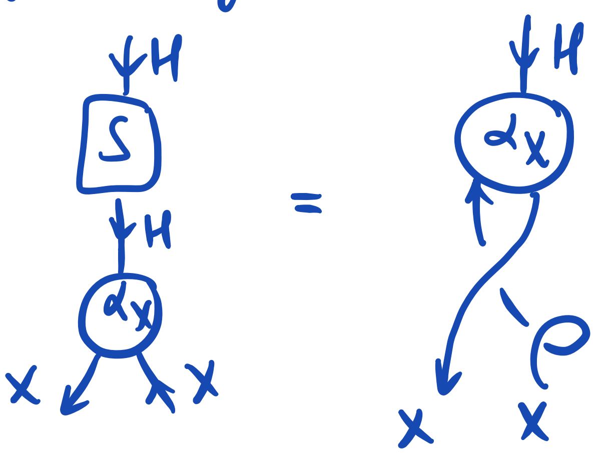
<u>The counite</u> is $\varepsilon : H \to \mathbb{1}$ defined by

$$\begin{tikzcd} X \otimes X^{*} \arrow[r, "\alpha_X"] \arrow[rr, bend right, "\tilde{\varepsilon}_X"'] & H \arrow[r, "\varepsilon"] & \mathbb{1} \end{tikzcd}$$

Graphically:  defined by



<u>The antipode</u> is $S : H \to H$ defined by

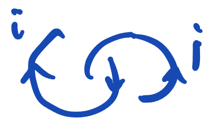


Coends and ends exist in nonsemisimple categories too. But we will not discuss it here. The standard notation is

$$H = \int^{X \in \mathcal{C}} X \otimes X^*$$

Basic propreties of MTC's. II

Let $\mathcal{C}$ be a modular tensor category,

i.e. $S_{ij} = \mathrm{inv}\Big($  $\Big)$ is nondgenerate

and $\sum_i d_i^2 \neq 0$.

We already proved:

1) $N_{ij}^{k} \overset{\text{def}}{=} \dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathcal{C}}(V_k, V_i \otimes V_j)\right)$

$$\underset{\text{braiding}}{N_{ij}^{k} = N_{ji}^{k}} \underset{\substack{\text{partial}\\ \text{dualization}}}{=} N_{ik^*}^{j^*} \underset{\text{total dualization}}{=} N_{i^*j^*}^{k^*}, \quad V_{i^*} = (V_i)^*$$

Note that N_{ij}^{k} are structural constants of the Grothendick ring $[\mathcal{C}]$:

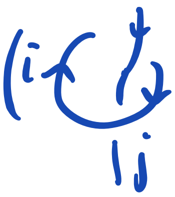
$$[V_i][V_j] = \sum_k N_{ij}^{k}\,[V_k]$$

2) $S_{i^*j^*} = S_{ij}$, $S_{ij} = S_{ji}$

3) $d_i d_j = \sum_k N_{ij}^{k} d_k$

The mapping $[V_i] \mapsto d_i$ can be regarded as a representation of $[\mathcal{C}]$

4) $$S_{ij} = \sum_k N_{ij}^k d_k v_k v_i^{-1} v_j^{-1}$$

5) $$\sum_i v_i^{\pm 1} \, \mathrm{inv}\Big($$  $$\Big) = p^{\pm} v_j^{\mp 1} \downarrow j$$

or $$\sum_i v_i^{\mp 1} S_{ij} \frac{1}{d_j} = p^{\pm} v_j^{\mp 1}$$

Define $|I| \times |I|$ matrices N_i, C

$(N_i)_k^j = N_{ik}^j$, $C_j^i = \delta_{i,j^*}$ and $x_i \in \mathbb{C}^{|I|}$

$(x_i)_j = \frac{1}{d_i} S_{ij}$ (assuming $d_i \neq 0$, $i \in I$)

Note that since $* : I \to I$ is an involution, $C^2 = \mathrm{Id}$.

<u>Proposition</u> The following identities hold:

1) $$N_i N_j = \sum_k N_{ij}^k N_k$$

2) $$N_i^t = C N_i C$$

3) $N_i \chi_j = S_{ij} \chi_j$

4) $C \chi_i = \chi_{i^*}$

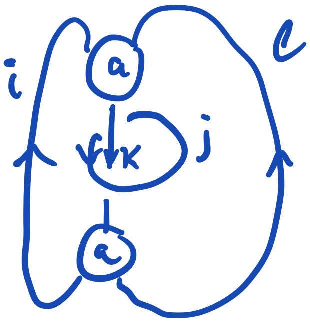
5) $d_i = S_{0i}$

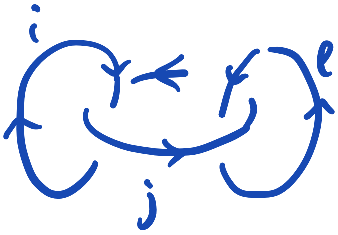
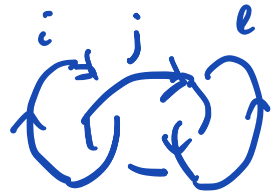
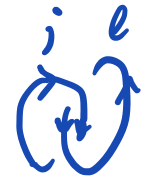
<u>Proof</u>. 1) $\sum_k N_{ik}^{l} N_{jm}^{k} = \sum_k \dim\big(\mathrm{Hom}_{\mathcal{C}}(V_l, V_i \otimes V_k) \otimes \mathrm{Hom}_{\mathcal{C}}(V_k, V_j \otimes V_m)\big) = \dim\big(\mathrm{Hom}_{\mathcal{C}}(V_l, V_i \otimes V_j \otimes V_m)\big)$

decomposing $V_j \otimes V_m$ into the direct sum of simple objects we obtain the desired identity

2) We already proved that $N_{ik}^{j} = N_{ij^*}^{k^*}$ since $\mathrm{Hom}(V_j, V_i \otimes V_k) \simeq \mathrm{Hom}(V_k^*, V_i \otimes V_j^*)$.

3)

$(N_i \chi_j)_l = \sum_k N_{il}^{k} \frac{S_{jk}}{d_j} = \sum_k \frac{1}{d_j}$  $=$

$= \frac{1}{d_j}$  $= \frac{1}{d_j}$  $= \frac{S_{ij}}{d_j}$ 

$$= \frac{S_{ij}}{d_j} S_{je} = S_{ij} (\chi_j)_e$$

4) $(C\chi_i)_j = \frac{1}{d_i} S_{ij^*} = \frac{1}{d_{i^*}} S_{i^*j} = (\chi_{i^*})_j$

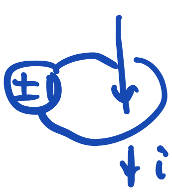
5) $d_i =$  $= S_{0i}$ by def. of S_{ij} ∎

Proposition The matrices S_{ij}, $T_{ij} = \delta_{ij} v_i$, $C_{ij^*} = \delta_{ij^*}$ satisfy relations

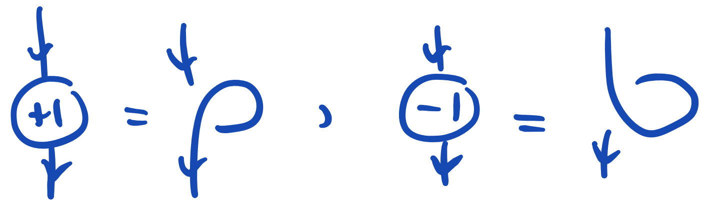
$$p^+ T^{-1} S T^{-1} = STS$$

$$p^- TST = ST^{-1}SC$$

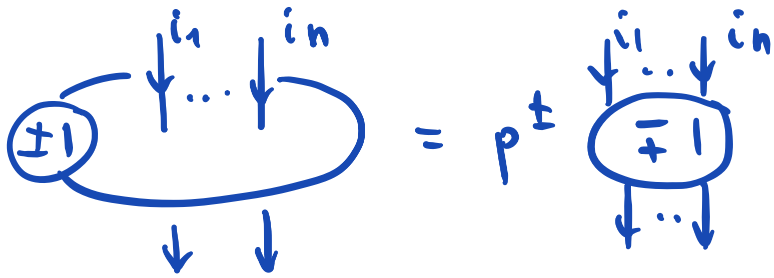
Proof. We already know that

 $= p^{\pm}$ , here — $= \sum_i d_i \xrightarrow{i}$

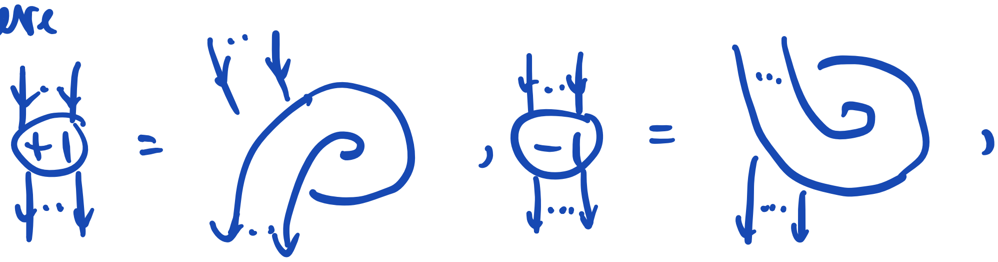
is the Kirby color and



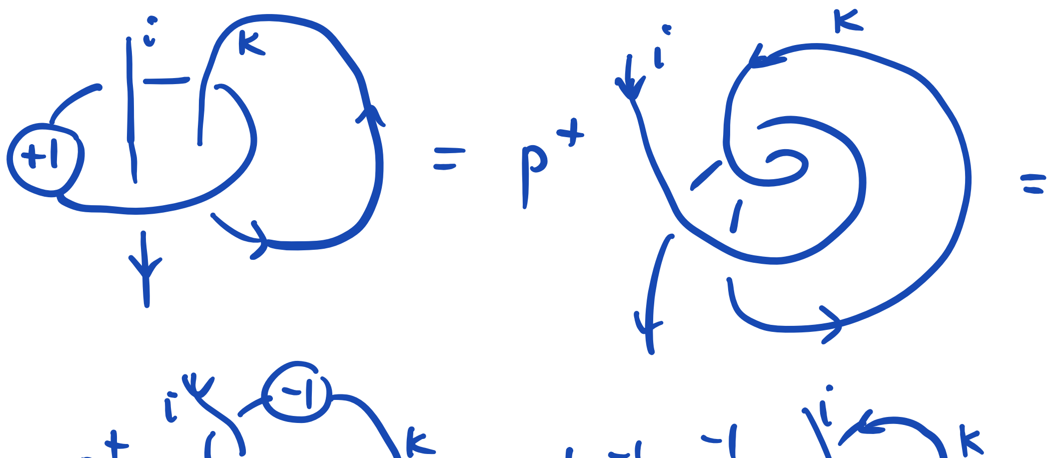
are twists of framing. From here follows that

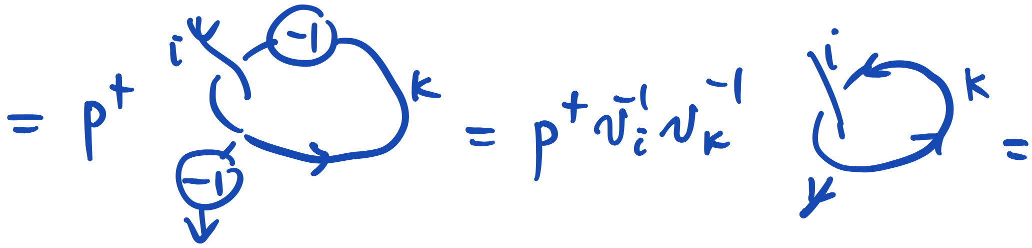


where



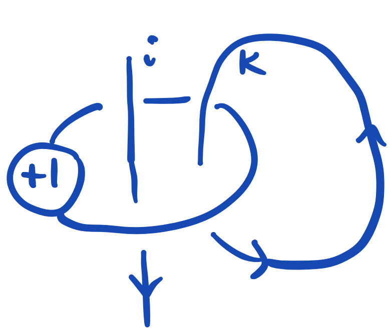
As a consequence, we have:

(*)


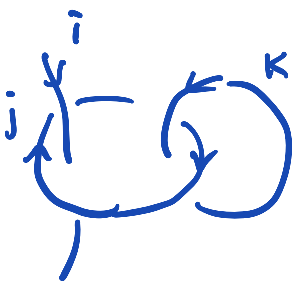


$$= p^+ v_i^{-1} S_{ik} v_k^{-1} d_i^{-1} \downarrow^i = \frac{p^+}{d_i}(T^{-1} S T^{-1})_{ik} \downarrow^i$$

Now, evaluate $(*)$ by definition of Kirby color:

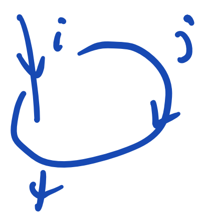


$$= \sum_j d_j v_j$$



$$=$$

$$= \sum_j d_j v_j \frac{S_{jk}}{d_j}$$



$$= \sum_j v_j S_{jk} S_{ij} \frac{1}{d_i} \downarrow^i$$

$$= (STS)_{ik} \frac{1}{d_i} \downarrow$$

Thus, $STS = p^+ T^{-1} S T^{-1}$.

Similarly, changing $+1$ twist to -1 we obtain the second identity.

<u>Corollary</u> $S^2 = p^+ p^- C$

Proof. From the proposition above we have:

$$p^+ T^{-1} S = STST$$

$$p^- STST = S^2 T^{-1} S C$$

Thus,

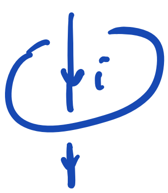
$$p^+ p^- T^{-1} S = S^2 T^{-1} S C$$

but $SC = CS$, $TC = CT$, therefore

$$p^+ p^- = S^2 C$$

Now use $C^2 = \mathrm{Id}$.

Proposition The following hold:

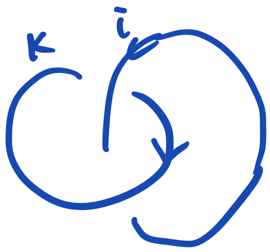
(i)  $= p^+ p^- \delta_{i,0}$ 

(ii) $p^+ p^- = \sum_i d_i^2$

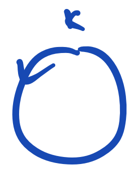
(iii)  $= \delta_{ij} \dfrac{p^+ p^-}{d_i}$ 

Proof. (i)

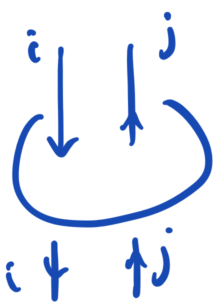
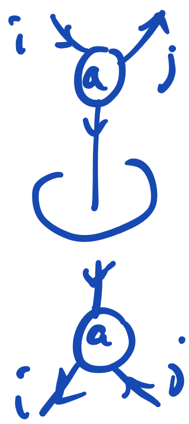
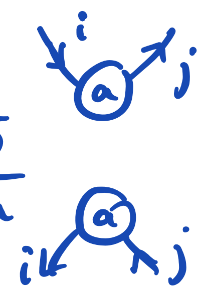
Let us close the i-stand

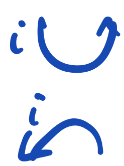
$\sum_k d_k$  $= \sum_k d_k S_{ki} = \sum_k S_{0k} S_{ki}$

$= (S^2)_{0i} = p^+ p^- (C)_{0i} = p^+ p^- \delta_{i,0}$

(ii) Let $i=0$ in (i), $p^+ p^- = \sum_k$  $d_k =$

$= \sum_k d_k^2$

(iii)  $= \sum_{k,a}$  $= p_+ p_- \sum_a$  $=$

$= p_+ p_- \frac{1}{d_i}$  δ_{ij} ,

orthonormal projector

$V_i \otimes V_j \to \mathbb{1}$