Lect 13

Last time:

- Surgery on S^3 along a framed link with an attachment map.
- Standard surgery on S^3 along a framed link.
- Kirby's theorem
- Invariants of closed 3-manifolds from invariants of links.

Invariants of 3-manifolds with ribbon graphs

Now we will extend the construction from the previous lecture to derive invariants of 3-manifolds with ribbon (framed) graphs.

Assume that we have a framed link L in $S^3 \setminus \Gamma$ where Γ is a framed graph. In other words we have $L \sqcup \Gamma \subset S^3$.

Performing standard surgery on L we obtain a pair $(M_L, \Gamma \subset M_L)$. Invariants of such pairs can be constructed similarly to invariants of closed manifolds.

Choose a coloring of Γ by a modular tensor category $\mathcal{C}$. Let

$$F(L_c \sqcup \Gamma_c \subset S^3)$$

be the invariant of a framed $\mathcal{C}$-colored graph $L_c \sqcup \Gamma_c \subset S^3$ that we constructed earlier. Define

$$\tau(\Gamma_c \subset M_L) = \mathcal{D}^{-|L|} \left(\frac{p_+}{p_-}\right)^{\frac{\sigma(L)}{2}} F(\Gamma_c \sqcup L \subset S^3)$$

where

$$F(\Gamma_c \sqcup L \subset S^3) = \sum_{j_1 \cdots j_m \in I} d_{j_1} \cdots d_{j_m} F(\Gamma_c \sqcup L_{j_1 \cdots j_m})$$

Here Γ_c is a $\mathcal{C}$-colored oriented framed graph, connected components of L are colored by $j_1 \cdots j_m$ and oriented by after summation the result does not depend on the orientation.

Now, after a short reminder of what a topological quantum field theory means functorially, we will construct a 3d TQFT using surjery and invariants of 3-manifolds with framed graphs constructed above.

Topological quantum field theory

We already discussed a functorial framework of quantum field theory. Now let us focus on topological quantum field theory in d dimensions.

The category of space times in this case is the category of cobordisms.

Objects in this category are $(d-1)$-dimensional manifolds, possibly with extra topological structure. In particular, we assume they are oriented.

Morphisms in this category are

d-dimensional cobordisms. For example morphisms between two $(d-1)$-manifolds N_1 and N_2 are d-manifolds M s.t. $\partial M = \overline{N_1} \sqcup N_2$ where the orientation of N_2 agree with M and the orientation of $\overline{N_1}$ is opposite to that of M.

A d-dimensional TQFT consists of the following data:

(i) To each $(d-1)$-dimensional closed oriented manifold N
we assign a vector space $H(N)$
with the following properties

(a) to each homeomorphism $f: N \to N'$
we assign a linear isomorphism
$f_*: \mathcal{H}(N) \to \mathcal{H}(N')$

(b) $\mathcal{H}(\bar{N}) \simeq \mathcal{H}(N)^*$

(c) $\mathcal{H}(\emptyset)$ is 1-dimensional, but an isomorphism $\mathcal{H}(\emptyset) \simeq \mathbb{C}$ does have to be fixed.

(d) $\mathcal{H}(N_1 \sqcup N_2) \simeq \mathcal{H}(N_1) \sqcup \mathcal{H}(N_2)$

(ii) To each d-dimensional oriented manifold we assign a vector

$Z(M) \in \mathcal{H}(\partial M)$

with the following properties

(a) If $f: M \xrightarrow{\simeq} M'$ is a homeomorphism and $(f|_{\partial M})_* : \mathcal{H}(\partial M) \to \mathcal{H}(\partial M')$ is the linear isomorphism corresponding to restriction of f to ∂M, then

$$(f|_{\partial M})_*(Z(M)) = Z(M')$$

(b) Assume $\partial M = N_1 \sqcup N_2 \sqcup N_3$ and $f: N_1 \xrightarrow{\sim} \overline{N_2}$ is a homeomorphism. Let M_f be the result of gluing N_1 to N_2, i.e. the result of identifying N_1 with $\overline{N_2}$ via f. Then $Z(M_f)$ should be the image of $Z(M)$ with respect to the linear mapping

$$\mathcal{H}(N_1) \otimes \mathcal{H}(N_2) \otimes \mathcal{H}(N_3) \xrightarrow{f_* \otimes id \otimes id}$$

$$\mathcal{H}(N_2)^* \otimes \mathcal{H}(N_2) \otimes \mathcal{H}(N_3) \xrightarrow{\langle \cdot, \cdot \rangle \otimes id} \mathcal{H}(N_3)$$

(c) If $M = I \times N$, $\partial M = N \sqcup \overline{N}$

then $\mathcal{H}(\partial M) \simeq \mathcal{H}(N) \otimes \mathcal{H}(N)^* \simeq \mathrm{End}(\mathcal{H}(N))$

and we require

$$Z(I \times N) = id_{\mathcal{H}(N)} \qquad (T)$$

<u>Remark 1</u> If M is a mapping cylinder

s.t. $\partial M = \bar{N} \sqcup N'$ with the homeomorphism $g: M \to I \times N$, $g|_N = id_N$, $g|_{N'}: N \to N'$ then

$$Z(M) \in \mathcal{H}(\bar{N}) \otimes \mathcal{H}(N') \simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(N), \mathcal{H}(N'))$$

and $\quad Z(M) = (g|_{N'})_*$

<u>Remark 2</u> The last property $Z(I \times N) = id_{\mathcal{H}(N)}$ is the essence of the topological nature of TQFT. If a nontopological quantum system where $I = [t_1, t_2]$ is a time interval we would have

$$Z(I \times N) = \exp\left(i \frac{t_2 - t_1}{\hbar} \hat{H}\right) \in \mathrm{End}(\mathcal{H}(N))$$

Here $\hat{H}$ is the Hamiltonian of the system acting on the space of boundary states. In the topological case $\hat{H} = 0$, i.e. the dynamics is trivial.

This structure of a TQFT has an immediate consequence.

<u>Corollary</u>. If we want to define the TQFT on all possible d-manifolds and in particular on $S^1 \times N$, then

$$Z(S^1 \times N) = \mathrm{tr}_{\mathcal{H}(N)}\left(Z(I \times N)\right) = \dim(\mathcal{H}(N))$$

i.e. we should have

$$\dim(\mathcal{H}(N)) < \infty$$

for all N.

<u>Remark</u>. We can as to TQFT to be defined on a subset of space times, for example, only for cylinders. Then we will not be restricted to finite dimensional $\mathcal{H}(N)$.

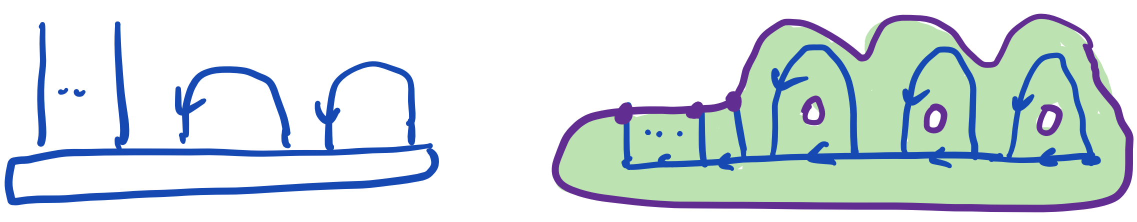
The data described above define the functor from the category of d-cobordisms to the category of vector spaces.

Modular tensor categories and 3-dimensional TQFT

Let us start with the description of the category of cobordisms.

Objects are compact oriented surfaces with puncture and with an extra structure called parametrization.

Define the standard surface $\Sigma^{st}_{g,n}$ as the boundary of a tubular neighborhood of a ribbon "flat graph" $\Gamma_{g,n}$ with g handles and n "loose ends, see Fig. (*)



Graph $\Gamma_{g,n}$ has one vertex with a coupon structure (Fig. on the left). It can be regarded as collapsing horizontal edges of the graph on the right Fig. along horizontal edges.

Def. A pair $(\Sigma_{g,n}, f)$ where f is a homeomorphism $f: \Sigma^{st}_{g,n} \to \Sigma_{g,n}$ is called a *parametrized* surface.

Now we will constuct 3 dim TQFT presenting 3-manifolds with parametrized components of boundary as a surgery on $S^3 \setminus \Gamma$.

First let us define the space $\mathcal{H}(\Sigma^{st}_{g,n})$ corresponding to a parametrized surface. Define the coloring of $\Gamma_{g,n}$ in the same way as it would do to define invariant of $\Gamma_{g,n} \subset S^3$:

- To each edge assign an object of $\mathcal{C}$. To internal edges assign simple objects
- To each vertex assign a morphism between objects corresponding to edges.

Define the space $H(\Sigma^{st}_{g,n})$ as a complex vector space with the basis parametrized by colorings of $\Gamma_{g,n}$ with fixed coloring $V_1, .., V_n$ of external edges

$$H(\Sigma^{st}_{g,n}; V_1, \dots, V_n) = \bigoplus_{\substack{\text{internal} \\ \text{colors}}} \mathbb{C}\, \Gamma_{g,n}(\text{colored})$$

Now let us give a basis free description of this space.

Define the <u>coend object</u> in our semisimple category as $H = \bigoplus_{i \in I} V_i \otimes V_i^*$ where

V_i are simple objects. Then

$$H(\Sigma^{st}_{g,n}; V_1, .., V_n) = Hom_{\mathcal{C}}(\mathbb{1}, V_1 \otimes \cdots \otimes V_n \otimes H^{\otimes g})$$

Proof is a h/w.

TQFT with no marked points on surfaces.

In the nex few lectures we will focus on TQFT with no marked points.

In this case

$$H(\Sigma^{st}_g) = Hom_{\mathcal{C}}(\mathbb{1}, H^{\otimes g})$$

Now, for each compact, oriented 3-manifold M with parametrized boundary

$$\partial M = \bigsqcup_{\alpha} \Sigma^{st}_{g_\alpha}$$

define the vector

$$\tau(M) \in \bigotimes_{\alpha} \mathcal{H}(\partial_\alpha M)$$

as follows.

Step 1. Represent M is a result of a result of a standard surgery on $S^3 \setminus \bigsqcup_\alpha \Gamma_{g_\alpha}$ along a framed link $L \subset S^3 \setminus \bigsqcup_\alpha \Gamma_{g_\alpha}$.

This is always possible and if L is Kirby equivalent to L', then corresponding 3-manifolds a homeomorphic via the homeomorphism acting trivially on the boundary.

Step 2. Define the vector $\tau(M_L)$

by its components in the basis of

$\bigotimes_\alpha \mathcal{H}(\Sigma^{st}_{g\alpha})$ corresponding to coloring of

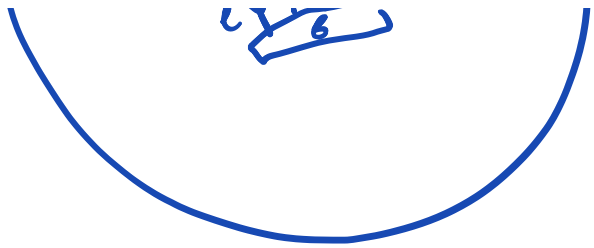
graphs $\Gamma_{g\alpha}$

$$\tau(M_L)_{\text{coloring}} = \mathrm{inv}\left(\bigsqcup_\alpha \Gamma_\alpha(\text{colored}) \sqcup L \subset S^3\right)$$

$$D^{-|L|}\left(\frac{p_+}{p_-}\right)^{\sigma(L)},$$

Here $\mathrm{inv}(\Gamma_c \sqcup L \subset S^3)$ is the invariant of a colored graph Γ_c and a link L with Kirby color in S^3.

An example of $\Gamma_c \sqcup L \subset S^3$:



$$a \in \mathrm{Hom}(\mathbb{1}, V_j \otimes V_j^* \otimes V_i \otimes V_i^*),$$

$$b \in \mathrm{Hom}(\mathbb{1}, V_l \otimes V_l^* \otimes V_k \otimes V_k^*),$$

Before proving that these data $\tau(M) \in \mathcal{H}(\partial M)$ define a 3dimensional TQFT let us review some properties of modular tensor categories.

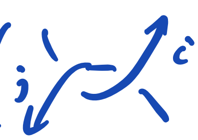
Coends and their properties. 1.

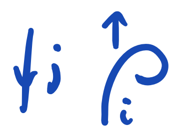
Let $\mathcal{C}$ be a modular tensor category. The object $H = \bigoplus_{i \in I} V_i \otimes V_i^*$ is called coend of $\mathcal{C}$.

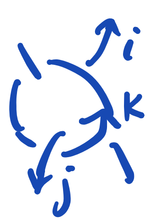
As we will see later, this object has

a natural structure of a Hopf algebra in the category $\mathcal{C}$ and a counterpart object $H_c = \bigoplus_i V_i^* \otimes V_i$, the end, with similar structure.

Now consider $\mathrm{End}_{\mathcal{C}}(H)$ and special elements $S, T, I : H \to H$ defined by their components mapping $V_j \otimes V_j^* \subset H$ to $V_i \otimes V_i^* \subset H$ as follows:

$\beta_i S \alpha_j = \mathrm{inv}\Big($  $\Big)\, d_i$

$\beta_i T \alpha_j = \mathrm{inv}\Big($  $\Big)\, \delta_{ij} = \delta_{ij}\, v_i\, \mathrm{inv}\big(\downarrow^i \uparrow^i\big)$

$\beta_i I \alpha_i = \sum_K \mathrm{inv}\Big($  $\Big)\, d_K d_i$

Note that for any object X spaces

$$\mathcal{H}_X^L = \mathrm{Hom}_{\mathcal{C}}(X, H), \quad \mathcal{H}_X^R = \mathrm{Hom}(H, X)$$

are left and right $\mathrm{End}_{\mathcal{C}}(H)$ modules respectively

Clearly, there are natural linear isomorphisms

$$\mathcal{H}^L_X \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, H \otimes X^\vee)$$

$$\mathcal{H}^R_X \simeq \mathrm{Hom}_{\mathcal{C}}(X^\vee \otimes H, \mathbb{1})$$

Thus

$$\mathcal{H}(\Sigma^{st}_{g,n}, V_n^\vee, \ldots, V_1^\vee) \simeq \mathcal{H}^L_{V_1 \otimes \cdots \otimes V_n}$$

and

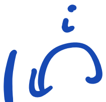
$$\mathcal{H}(\Sigma^{st}_{g,n}, V_n^\vee, \ldots, V_1^\vee)^* \simeq \mathcal{H}^R_{V_1 \otimes \cdots \otimes V_n}$$

The natural pairing

$$\mathcal{H}^L_X \otimes \mathcal{H}^R_X \to \mathrm{Hom}_{\mathcal{C}}(X, X)$$

which is just the composition of morphisms is the dualization when X is simple and therefore $\mathrm{Hom}_{\mathcal{C}}(X, X) = \mathbb{C}$.

Consider $\mathcal{H}^{L}_{\mathbb{1}}$ and $\mathcal{H}^{R}_{\mathbb{1}}$. We have a natural bases in these spaces

$$E_i = \mathrm{inv}\Big($$  $$\Big) \in \mathcal{H}^{L}_{\mathbb{1}}, \quad F_i = \mathrm{inv}\Big($$  $$\Big) \in \mathcal{H}^{R}_{\mathbb{1}}$$

with the pairing

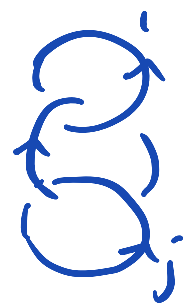
$$\langle F_i, E_j \rangle = \delta_{ij}\, d_i$$

Automorphisms $S, T, I \in \mathrm{End}_{\ell}(H)$ act on the basis E_i by matrices S_{ij}, T_{ij}, I_{ij}:

$$S(E_i) = \sum_{j \in I} S_{ji} E_j, \quad S_{ji} = \mathrm{inv}\Big($$  $$\Big)$$

$$T(E_i) = \sum_j T_{ji} E_j = v_i E_i, \quad T_{ij} = \delta_{ij} v_i$$

$$I(E_i) = \sum_j I_{ji} E_j,$$

$$I_{ij} = \sum_k d_k \,\mathrm{inv}\Big($$  $$\Big)$$

Next time we will prove

$$I_{ij} = \delta_{0j} D$$

where $D = \sum_{i \in I} d_i^2$.