Lecture 6

Path integral representation for $U_t(q,q')$

Feynman's idea, originated in the Wiener formula in probability theory is that the propagator can be written an "integral" over all parametrized paths connecting q and q' in time t.

$$U_t(q,q') = \int_{\substack{q(0)=q \\ q(t)=q'}} e^{\frac{i}{\hbar} S[\gamma]} \mathcal{D}\gamma \qquad (FI)$$

Here we assume $S[\gamma] = \int_0^t \left(\frac{m}{2} \dot{q}(\tau)^2 - V(q(\tau)) \right) d\tau$

How to formulate this idea in a mathematically meaningful way?

One way to do it is to define the formal asymptotical expansion as $\hbar \to 0$ by the analogy with finite dimensional oscillatory integrals. Let us make a digression into such integrals.

Consider

$$I_\hbar = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} f(x)} g(x)\, d^n x$$

Assume: $f(x)$ is smooth with unique nondegenerate critical point x_0 $(\det(f^{(2)}(x_0)) \neq 0)$

The integral is defined as analytical cont. from $\operatorname{Im} \hbar > 0$.

It is not difficult to prove the following:

$$I_\hbar \underset{\uparrow}{\cong} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} f(x_0 + \sqrt{\hbar}\, t)} \hbar^{\frac{n}{2}} d^n t$$

asymptotical equivalence

$$= \hbar^{\frac{n}{2}} e^{\frac{i f(x_0)}{\hbar}} \times$$

$$\times \int_{\mathbb{R}^n} e^{\frac{i}{2} \frac{\partial^2 f(x_0)}{\partial x^i \partial x^j} t^i t^j + \sum_{k \geq 3} h^{\frac{k}{2}-1} \frac{1}{k!} \sum_{i_1 \cdots i_k} \frac{\partial^k f(x_0)}{\partial x^{i_1} \cdots \partial x^{i_k}} t^{i_1} \cdots t^{i_k}} d^n t$$

Expanding this into power series in h we obtain a formal power series in h with coefficients given by integrals

$$J^{i_1 \cdots i_k} = \int_{\mathbb{R}^n} e^{\frac{i}{2} \frac{\partial^2 f(x_0)}{\partial x^i \partial x^j} t^i t^j} t^{i_1} \cdots t^{i_k} \, d^n t$$

If $\operatorname{Im}\left(\frac{\partial^2 f(x_0)}{\partial x^i \partial x^j} \right)$ is a positive definite matrix the integral converges and if k is even

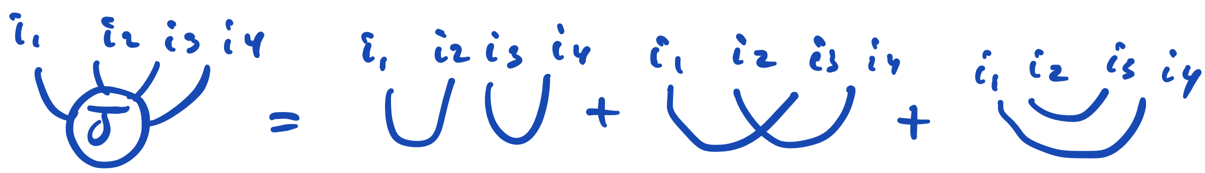
$$J^{i_1 \cdots i_k} = i^k \sum_{\substack{\text{perfect matchings} \\ \text{on } \{i_1 \cdots i_k\}}} G^{m_1 m_2} G^{m_3 m_4} \cdots G^{m_{k-1} m_k} \int_{\mathbb{R}^n} e^{\frac{i}{2}(f^{(2)}(x_0) t, t)} d^n t$$

where $G^{kl} = \left(\left(\frac{\partial^2 f(x_0)}{\partial x \, \partial x} \right)^{-1} \right)^{kl}$. $J^{\{i\}} = 0$ if k is odd. (proof $=$ h/w)

For example when $k=4$

$$J^{i_1 i_2 i_3 i_4} = i^2\left(G^{i_1 i_2} G^{i_3 i_4} + G^{i_1 i_3} G^{i_2 i_4} + G^{i_1 i_4} G^{i_2 i_3}\right)$$

or graphically:



From here we derive the asymptotical expansion of I_h as $h \to 0$ (prove it, h/w)

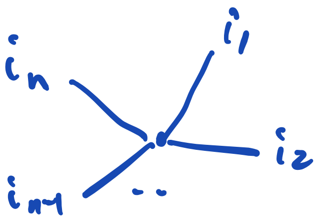
$$I_h \simeq (2\pi h)^{\frac{n}{2}} e^{\frac{i f(x_0)}{h}} \frac{1}{\left|\det\left(\frac{\partial^2 f(x_0)}{\partial x^i \partial x^j}\right)\right|^{1/2}}$$

$$e^{i\frac{\pi}{4}\operatorname{sign}(f^{(2)}(x_0))}\left(1+\sum_{\Gamma}(ih)^{-\chi(\Gamma)} F(\Gamma)\right)$$

Here Γ are graphs with valencies of vertices ≥ 3. Function $F(\Gamma)$ is defined as follows:

$$i \;\text{———}\; j \;\rightsquigarrow\; G^{ij} = \left(f^{(2)}(x_0)^{-1}\right)^{ij}$$

$$f^{(2)}_{ij} = \frac{\partial^2 f(x_0)}{\partial x^i \partial x^j},$$

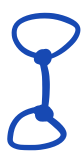


$$\rightsquigarrow \frac{\partial^n f(x_0)}{\partial x^{i_1} \ldots \partial x^{i_n}},$$

$$F(\Gamma) = \sum_{\{i\}} \prod_{\text{edges}} G^{i_{e_+} i_{e_-}} \prod_{\text{vertices}} f^{(n)}_{(x_0)}{}^{i_1 \ldots i_n} \cdot$$

$$\cdot \frac{1}{\mathrm{Aut}(\Gamma)},$$

Order 1 in $\hbar$ contributions:

$\frac{1}{2^3} F($  $) + \frac{1}{3!\,2!} F($  $) +$

$+ \frac{1}{2} F($  $)$

$$-\chi(\Gamma) = 2-1 = 3-2 = 2-1 = 1$$

$F($  $) = \sum f^{(3)}_{i_1 j_1 k_1} f^{(3)}_{i_2 j_2 k_2} G^{i_1 j_1} G^{i_2 j_2} G^{k_1 k_2}$

$$F(\ominus) = \sum f^{(3)}_{i_1 j_1 k_1} f^{(3)}_{i_2 j_2 k_2} G^{i_1 i_2} G^{j_1 j_2} G^{k_1 k_2}$$

$$F(8) = \sum f^{(4)}_{i_1 i_2 i_3 i_4} G^{i_1 i_2} G^{i_3 i_4}$$

Note that (prove it, hw)

$$1 + \sum_{\Gamma} (i\hbar)^{-\chi(\Gamma)} \frac{F(\Gamma)}{|\mathrm{Aut}(\Gamma)|} = \exp\Big(\sum_{\substack{\Gamma \\ \text{connected}}} (i\hbar)^{-\chi(\Gamma)} \frac{F(\Gamma)}{|\mathrm{Aut}(\Gamma)|} \Big)$$

Now let us define the integral (FI) mimicking the asymptotic of $I_\hbar$ as $\hbar \to 0$.

Let γ_0 be the critical point of $S[\gamma]$ and $s = \{x(\tau)\}_{\tau=0}^{t}$ be a path in $\mathbb{R}$ such that $x(0) = x(t) = 0$

$$S[\gamma_0 + \sqrt{\hbar}\, s] = S[\gamma_0] + \frac{\hbar}{2} \int_0^t \Big(\frac{m}{2} \dot{x}(\tau)^2 -$$

[illegible]

$$- V(q_0(\tau))\, x(\tau) \,] \, d\tau \, +$$

$$+ \sum_{k \geq 3} \frac{\hbar^{\frac{k}{2}}}{k!} \int_0^t V^{(n)}(q_0(\tau))\, x(\tau)^n \, d\tau$$

If the path integral would make sense we would have

$$\int e^{\frac{i}{\hbar} S[\gamma]} \mathcal{D}\gamma =$$

$$\int e^{\frac{i S(\gamma_0)}{\hbar} + \frac{i}{2}(x, A x) + \sum_{k \geq 3} i \frac{\hbar^{\frac{k}{2}}}{k!} \int_0^t V^{(n)}(q_0(\tau))\, x(\tau)^n d\tau} \mathcal{D}x$$

Gaussian path integral we can define by analogy with the finite dimensional case as

$$\int e^{\frac{i}{2}(x, Ax)} \mathcal{D}x = C \, \frac{1}{\sqrt{|\det_{reg}(A)|}}$$

where C is some infinite constant.

The determinant $\det_{reg}(A)$ does not make sense as the determinant, but there is a standard, so called ζ-regularization (see ...) which we assume here $\det_{reg}(A) = \det_\zeta(A)$.

The analog of Wick's theorem: for even k

$$\frac{\int e^{\frac{i}{2}(x,Ax)} x(\tau_1)\cdots x(\tau_k)\mathcal{D}x}{\int e^{\frac{i}{2}(x,Ax)}\mathcal{D}x} =$$

$$= i^k \sum_{\substack{m_1 \cdots m_k \\ \text{perfect matchings}}} G(\tau_{m_1},\tau_{m_2})\cdots G(\tau_{m_{k-1}},\tau_{m_k})$$

Here $G(\tau,\tau')$ is the Green function for A:

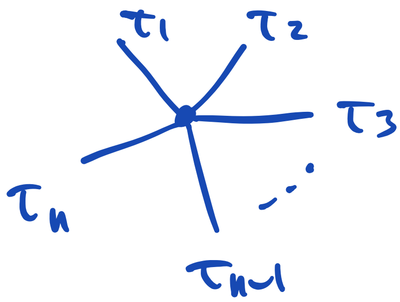
$$\begin{cases} \left(-\dfrac{d^2}{d\tau^2} - V''(q_0(\tau))\right) G(\tau,\tau') = \delta(\tau,\tau') \\ G(0,\tau') = G(t,\tau') = 0, \quad G(\tau,\tau') = G(\tau',\tau) \end{cases}$$

Thus, we can define (by the analogy with the asymptotical expansion of finite dimensional oscillatory integrals) the path integral as

$$\int_{\substack{q(0)=q' \\ q(t)=q}}^{\text{formal}} e^{\frac{i}{\hbar} S[\gamma]} \, \mathcal{D}\gamma = C \, \frac{e^{i\frac{S[\gamma_0]}{\hbar}}}{\sqrt{\det_\zeta (A_{\gamma_0})}} \cdot \left(1 + \sum_{\Gamma} (i\hbar)^{-\chi(\Gamma)} F(\Gamma)\right)$$

Feynman diagrams $F(\Gamma)$ are determined by the following building blocks

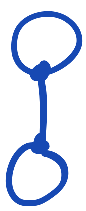
$$\tau \;\text{———}\; \tau' \rightsquigarrow G(\tau,\tau')$$



$$\rightsquigarrow \; V^{(n)}(q_0(\tau_i)) \prod_{j} \delta(\tau_j - \tau_{j+1})$$

(cyclic)

For example here is one order 1 term:

$F\Big($  $\Big) =$

$$\int_0^t\!\!\int V^{(3)}(q_0(\tau_1))\, V^{(3)}(q_0(\tau_2))\, G(\tau_1,\tau_1)\, G(\tau_2,\tau_2)\, G(\tau_1,\tau_2)\, d\tau_1\, d\tau_2$$

If the $G(\tau,\tau')$ is singular on the diagonal $\tau=\tau'$, expressions like this do not make sense. Fortunately in our case it is not

<u>Theorem 1</u> If $V^{(2)}(q)=0$, i.e.

$$-\frac{d^2}{d\tau^2}G_0(\tau,\tau') = \delta(\tau,\tau')$$

we have

$$G_0(\tau,\tau') = \begin{cases} \tau\,\frac{t-\tau'}{t}, & \tau<\tau' \\ \tau'\,\frac{t-\tau}{t}, & \tau>\tau' \end{cases}$$

<u>Thm 2</u>. As $\tau \to \tau'$

$$G(\tau,\tau') = G_0(\tau,\tau') + O(|\tau-\tau'|)$$

<u>Corollary</u>. Each integral in $F(\Gamma)$ converges.

Another very important result was obtained by Gelfand and Yaglom:

<u>Theorem</u>

$$\left|\det_{\zeta}(A_{\gamma_0})\right| = \frac{1}{\left|\frac{\partial^2 S_t(q,q')}{\partial q\,\partial q'}\right|}$$

Taking into account that $S[\gamma_0] =$

$= S_t(q,q')$, we obtain

$$\int_{\substack{q(0)=q'\\q(t)=q}}^{\text{formal}} e^{\frac{i}{\hbar} S[\gamma]} \mathcal{D}\gamma = C \frac{e^{\frac{i}{\hbar} S_t(q,q')}}{\sqrt{\left|\frac{\partial^2 S_t(q,q')}{\partial q \, \partial q'}\right|}} \cdot \Big(1 + \sum_{\Gamma} (i\hbar)^{-\chi(\Gamma)} F(\Gamma)\Big)$$

But we already know that when $C = (2\pi h)^{-\frac{1}{2}} e^{-i\frac{\pi}{4}}$ the leading term is exactly the semiclassical asymptotic of $U_t(q,q')$. Thus, the semiclassical path integral formula asserts:

$$U_t(q,q') = \int_{q(0)=q',\, q(t)=q}^{\text{formal}} e^{\frac{i}{\hbar} S[\gamma]} \mathcal{D}\gamma$$

$$= (2\pi h)^{-\frac{1}{2}} \frac{e^{\frac{i}{h} S_t(q, q') - i\frac{\pi}{4}}}{\sqrt{\left| \frac{\partial^2 S_t(q, q')}{\partial q \, \partial q'} \right|}} \cdot \qquad (*)$$

$$\Big(1 + \sum_{\Gamma} (ih)^{-\chi(\Gamma)} F(\Gamma)\Big)$$

where $F(\Gamma)$ are given by Feynman diagrams described above

<u>Theorem</u> (T. Johnoson-Freyd) The expression $(*)$ satisfies the Schrödinger equation (2a) and the semigroup law (2c) in all orders.