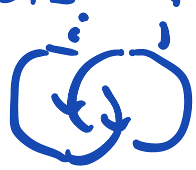
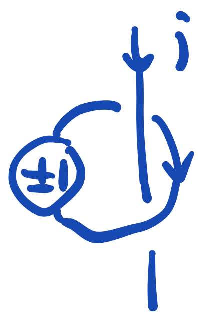
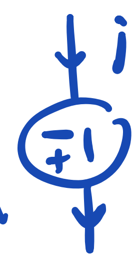
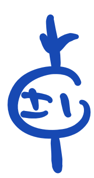
Lect 12

Last time:

- Modular tensor categories
  - semisimple, abelian ribbon, finitely many simple objects.
  - $S_{ij} = \mathrm{inv}\Big($  $\Big)$ is invertible.

Thm

$\mathrm{inv}\Big($  $\Big) = p^{\pm}\ \mathrm{inv}\Big($  $\Big)$

Here  $= \pm$ twist of framing,

$$\downarrow \; = \sum_j \underset{\dim(V_j)}{d_j} \ \text{''} \qquad\qquad \downarrow^{j} \; = \sum_j d_j \uparrow^{j^*}$$

$$= \sum_{j} d_j \uparrow^{j} \quad \text{because } d_{j^*} = d_j$$

$j^* \to j$

$V_j^* \simeq V_{j^*}$, $\quad * : I \to I$

$\uparrow$ set of simple objects.

$p^{\pm} = \sum_i d_i v_i^{\pm 1}$, inv() = v_j inv()

$\uparrow$ ribbon

<u>Modularity</u> $\longleftrightarrow SL_2(\mathbb{Z})$

Let $T_{ij} = v_i \delta_{ij}$, S_{ij} is as above

$C_{ij} = \delta_{ij^*}$, $\qquad \tilde{S}$

v_i, S_{ij} : $v_{i^*} = v_i$, $S_{i^*j^*} = S_{ij}$

- $\Rightarrow$ $\underline{TC = CT}$, $\underline{SC = C\tilde{S}}$
- $*$ is an involution $\Rightarrow$

$C^2 = I$, $\quad p^{\pm} = \sum_i d_i v_i^{\pm 1}$,

<u>Thm</u> $(*)$

- $(ST)^3 = p^+ S^2$
- $(ST^{-1})^3 = p^- S^2 C$

$\boxed{\cdot\ p^+ p^- = \underbrace{\sum_i d_i^2}_{"D^2"}}$

D = the dimension of $\mathcal{C}$.

• $s^2 = p^+ p^- C$ $\qquad \boxed{\left(\frac{p^+}{p^-}\right) \in S^1}$

<u>Proof.</u> (h/w) Bakalov, Kirillov.

$$SL_2(\mathbb{Z}), \quad \underset{t}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \quad \underset{s}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$s^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = c$$

$$st = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$(st)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(st)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = s^2 = c$$

s, t - generators of $SL_2(\mathbb{Z})$,

$s^2 = c$, $(st)^3 = s^2$, $c^2 = 1$ - def. relations

Claim. Then $(*)$ gives a proj. representation of $SL_2(\mathbb{Z})$ in $\mathbb{C}^N$, $N = |I|$

More generally: MTC $\to$ TQFT

$\Rightarrow$ gives a projective representation of the mapping class group $\mathcal{M}(\Sigma_g)$

$SL_2(\mathbb{Z}) = \mathcal{M}(\Sigma_1)$, $\mathcal{M}(\Sigma) = \mathrm{Diff}(\Sigma) / \mathrm{Diff}_0(\Sigma)$

torus

discrete

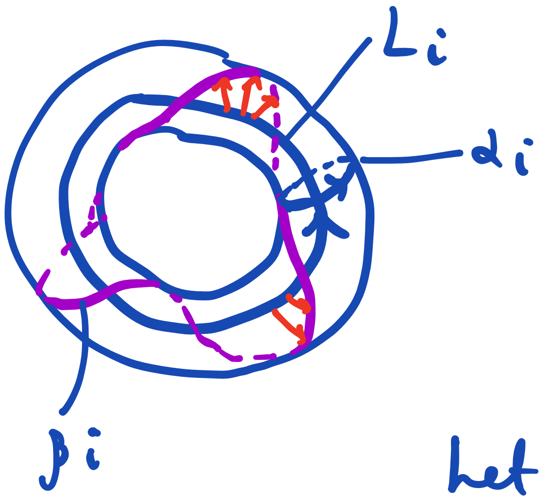
Invariants of closed 3-manifolds

Closed compact, connected 3-manifolds as a surgery along a framed link.

$L \subset S^3$ framed link.

L_i — connected components of L

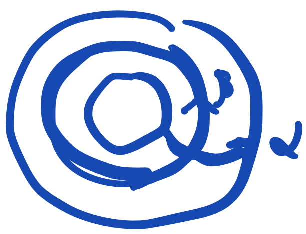
T_i is a tubular neighborhood of L_i



A basis in $H_1(\partial T_i)$

- α_i
- β_i is determined by the framing.

Let T_0 be a standard solid torus



Let $f_i : \partial T_i \to \partial T_0$ be a homeomorphism.

Facts: f_i extend to a homeomorphism of solid tori.

Def. $M_{L,f}$ 3-manifold, the surgery along L with attachment maps f_i is

$$\left(M \setminus \bigcup_{i=1}^{N} T_i^{int}\right) \cup_{f_i} \left(T_0^{N}\right)$$

$N = |L|$

Thm. Any closed orientable connected 3-manifold can be obtained as a surgery of S^3 along a framed link L.

Fix homeom. classes of f_i

assume, $f_i : \alpha \mapsto -\beta,\ \beta \mapsto \alpha$ ✓

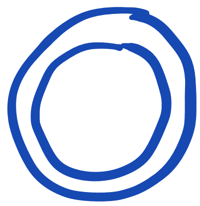
Fact. Such f_i exists and unique up to a homeom.

Def. M_L is the surgery on S^3 along L with these attachment maps.

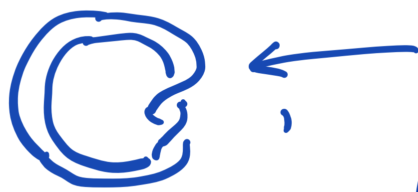
Remark. M_L does not depend on the orientation of components of L.

(change of orient. $\alpha \mapsto -\alpha,\ \beta \mapsto -\beta$ can be extended as a homeom. of T_0 $\Rightarrow$ does not change M_L)

Examples

(i) $L =$  no twist of framing

(h/w) Show that M_L (surgery on S^3 along L) $\simeq S^2 \times S^1$

(ii) $L =$ 

$(n=0)$

$\uparrow$

from the previous thm.

$$M_L \simeq S^3$$

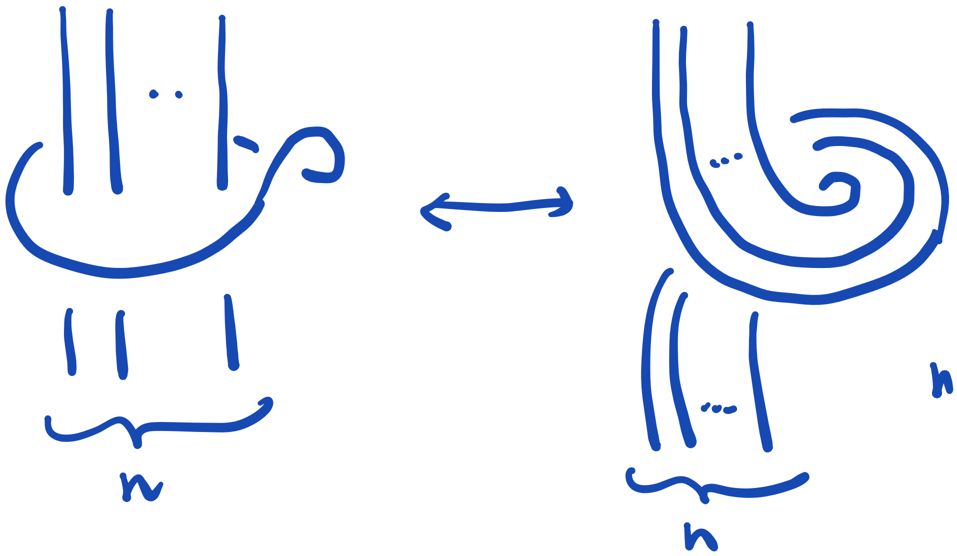
(h/w)

Q: which framed links give equivalent 3-manifolds?

Thm (Kirby) $M_L \simeq M_{L'}$ iff

surgery on S^3

L can be obtained from L' by a sequence of moves (Kirby moves)



$n = 0, 1, 2, \dots$

<u>Corollary</u> if $inv(L)$ is also invariant with respect to <u>the Kirby moves</u>, then it is an inv. of 3-manifolds

$$inv(M_L) \overset{def}{=} inv(L)$$

Consider the invariant of links

$$\tilde{\tau}(L) = D^{-|L|-1} \left(\frac{p_+}{p_-}\right)^{\frac{\sigma(L)}{2}} \times$$

$$\times F(L).$$

L is not oriented

$N = |L|$, ($\sigma(L)$ the signature of L will be defined later)

$$F(L) = \sum_{i_1 \cdots i_N \in I} d_{i_1} \cdots d_{i_N} \cdot \mathrm{inv}\left(\tilde{L} \text{ conn. components are colored by } i_1 \cdots i_N\right)$$

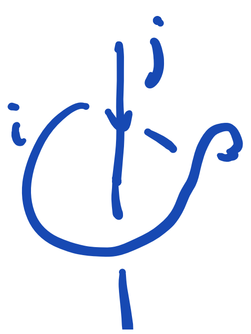
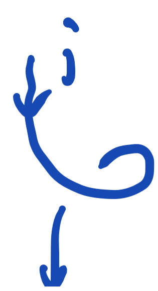
The one that we studied before

(i) $\tilde{L}$ - choose any orientation of L

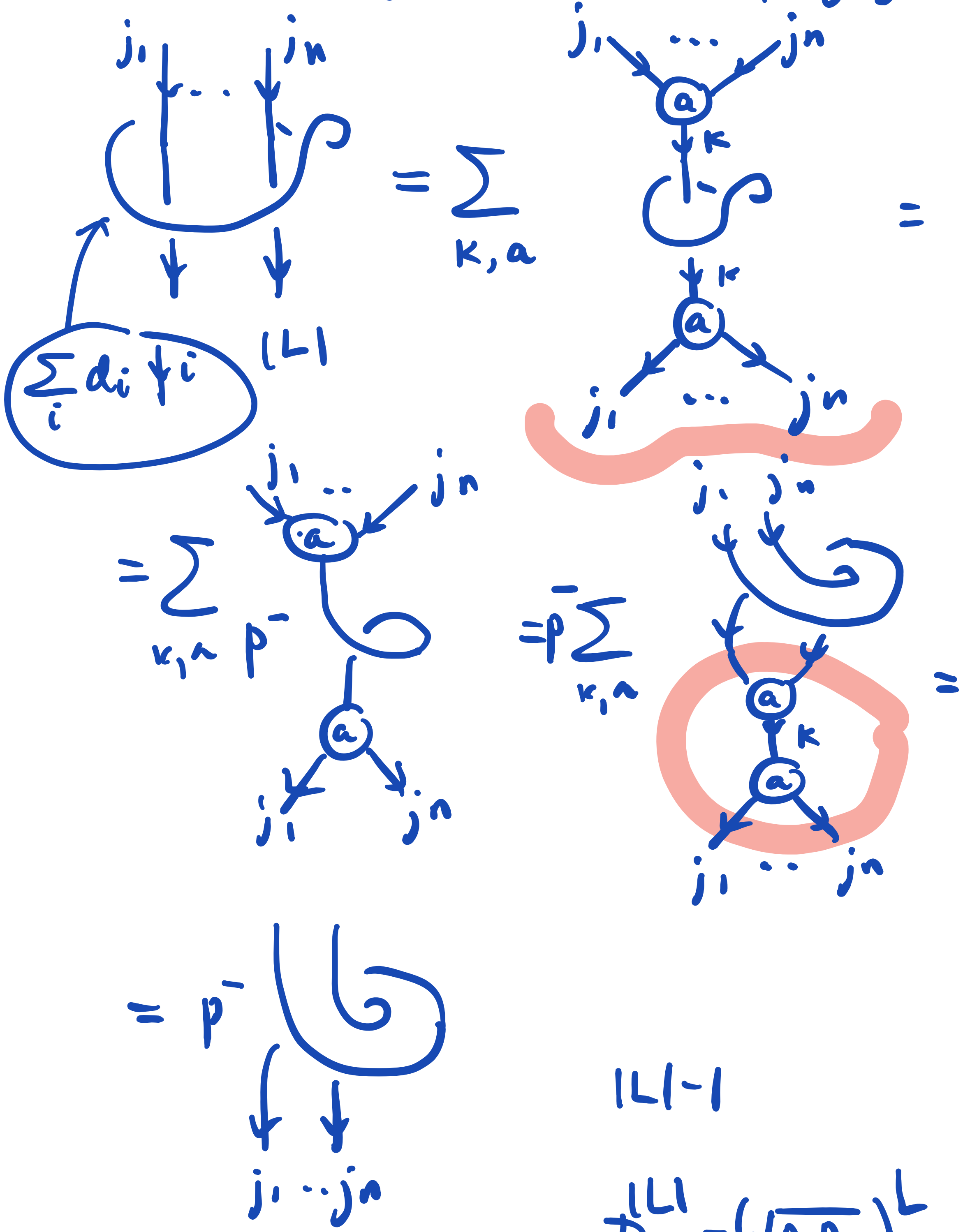
then the sum does not depend on the orientation.

$$\sum_i d_i \downarrow i = \sum_i d_i \uparrow i^* = \sum_j d_j \uparrow j \quad , \quad j = i^*$$

(ii)

$\sum_i d_i$  $= p^-$ 

$n=1$, Kirby invariance (proj.)



$|L|-1$

$$D^{|L|} = \left(\sqrt{p_+ p_-}\right)^L = \sqrt{p_+ p_-}\; D^{|L|-1}$$

Taking into account $(p^+p^-)^{\frac{|L|}{2}}$ factor we have projective multiplier in Kirby moves $\left(\frac{p_+}{p_-}\right)$

This is compensated by $\left(\frac{p_+}{p_-}\right)^{\sigma(L)}$

<u>Thm</u>. $\tilde{\tau}$ is invariant with respect to Kirby moves $\Rightarrow$

$$\tau(M_L) = \tilde{\tau}(L)$$

Witten-Reshetikhin-Turaev

<u>Example of</u> τ for MTC related to $\mathbb{Z}_n$.

$\mathcal{C}_n$, simple objects $V_0, \dots, V_{n-1}$

$$V_i \otimes V_j = V_{i+j \bmod n} = V_j \otimes V_i$$

$$c_{jj'} = \exp\left(\frac{i\pi}{n} jj'\right)$$

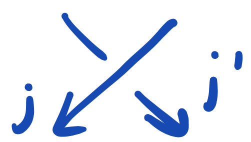
$$v_j = \exp\left(\frac{2i\pi}{n} j^2\right)$$

meta

h/w

$d_j = 1$,

$$\tau_n(L) = (p^+p^-)^{-\frac{|L|}{2}-\frac{1}{2}} \left(\frac{p^+}{p^-}\right)^{\frac{\sigma(L)}{2}} \cdot \sum_{j_1\cdots j_N} \exp\left(\frac{i\pi}{n}\sum_{\alpha,\beta=1}^{N} j_\alpha L_{\alpha\beta}\, j_\beta\right)$$



$\to \exp\left(\frac{i\pi}{n} j j'\right)$

$D = n = \sum_{i=0}^{n-1} d_i^2$

$p^+ = \sum_{k=0}^{n-1} \exp\left(\frac{\pi i k^2}{n}\right)$

$L_{\alpha\beta}$ = the linking number between conn. comp. α and β of L (Gauss invariant)

$L_{\alpha\alpha}$ = the winding number of the framing of α-th component

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Turaev-Viro invariants $\leftarrow$ triangulation

$$|RT|^2 = \underset{\uparrow \text{surgery}}{RT(M)}\, RT(-M) = \underset{\uparrow \text{triang}}{TV(M)}$$

$$\Gamma \subset S^3$$
linked
$$L \subset$$

surgery on S^3 along L $\Rightarrow$

$$\Rightarrow \quad \tilde{\Gamma} \subset M_L$$

This way we obtained $(\tilde{\Gamma} \subset M_L)$

Similarly to above $inv(\Gamma \subset M_L)$

$$\tilde{\tau}(\Gamma \sqcup L) = D^{-|L|-1} \left(\frac{p_+}{p_-}\right)^{\frac{\sigma(L)}{2}} \times$$
$(k_1 \ldots k_N, b_i)$

$$\times F(L).$$

L is not oriented

$N = |L|$, $\sigma(L)$ the signature of L, will be defined later

$$F(L) = \sum_{i_1 \ldots i_N \in I} d_{i_1} \cdots d_{i_N} \cdot$$

$$\cdot\, inv\left(\tilde{L}\ \text{conn. components are colored by } i_1 \ldots i_N \sqcup \Gamma(k, b)\right)$$

The one that we studied before

<u>Thm</u>. This gives invariant:

$$\tau(\Gamma \subset M_L)$$

an inv. of M_L with $\partial M_L = \partial(T_\Gamma)$

$$\boxed{q = \exp\left(i \frac{\pi}{k+g}\right)}$$

$k = 1, 2, \ldots$

k – the level in Ch.S (WZW)

g = the dual Coxeter

$$q = \exp\left(i \frac{\pi m}{k+g}\right)$$

?

not the Ch.S. (with compact G)

WZW for fractional k?
