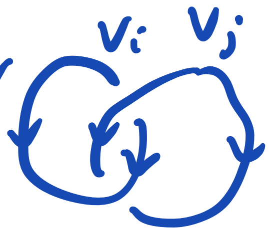
Lect 11

Last time: • Functorial aspect of a quantum field theory and a few examples.

The plan:

- Modular Tensor Category
- Examples
- invariants of closed 3-manifolds

Def. A category $\mathcal{C}$ is a MTC

- $\mathcal{C}$ is a ribbon category
- Semisimple. Finitely many isom. classes of simple objects.
- $S_{ij} = \mathrm{inv}\Big($ 

$\Big)$ this matrix is invertible.

Note $V_i \to V_i^*$ if V_i is simple then V_i^* is also simple $\Rightarrow V_i^* \cong V_{i^*}$

for some $i^* \in I$ (I = the set of isom. classes of simple objects)

$\Rightarrow$ duality (rigidity) gives $*: I \to I$ involution ($V_i^{**} \simeq V_i$ for ribbon).

General properties later.

Examples: 1) $q = e^{\frac{i\pi}{k}}$, (can be $e^{\frac{i\pi m}{k}}$ $m;k$ - rel. prime & m,k - odd)

Define the category $\mathcal{C}_k$ as:

- simple objects $\underset{\overset{\shortparallel}{V_0}}{\mathbb{1}}, V_1, \dots, V_{k-1}$

$$\mathrm{Hom}(V_i, V_j) \simeq \mathbb{C}\,\delta_{ij}$$

- $V_i \otimes V_j \overset{\mathrm{def}}{=} V_{i+j \,(\mathrm{mod}\, k)}$

$$(V_i \otimes V_j) \otimes V_\ell = V_i \otimes (V_j \otimes V_\ell)$$

$V_0 = \mathbb{1}$ is an identity object

$V_i^* = V_{-i}$, $\quad e: V_i^* \otimes V_i \to \mathbb{1}$,
$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{\mathbb{1}} \quad 1 \in \mathrm{Hom}(\mathbb{1}, \mathbb{1}) = \mathbb{C}$

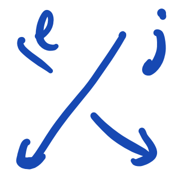
$i: \mathbb{1} \xrightarrow[1]{} V_i \otimes V_i^* = \mathbb{1}$

- **Braiding:** $i+j(K) = i+j \mod (K)$,

$$c_{ij} : V_i \otimes V_j \to V_j \otimes V_i = V_{i+j(K)} = V_i \otimes V_j$$

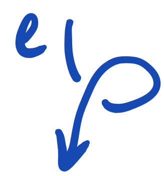
$$c_{ij} \in \mathrm{Hom}\left(V_{i+j(K)}, V_{i+j(K)}\right) \simeq \mathbb{C}$$

$$c_{\ell j} = \exp\left(\frac{i\pi}{K}\ell j\right) \leftarrow$$



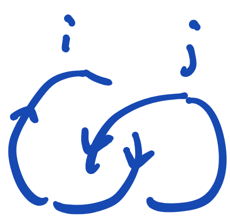
This gives the ribbon structure:

$$v_\ell = e^{i\pi\frac{\ell^2}{K}} \leftarrow$$



The S-matrix:

$$S_{ij} = \mathrm{inv}\Big($$



$$\Big)$$

$$S_{j'j} = c_{j'j}\, c_{jj'} = \exp\left(\frac{2\pi i}{K} j' j\right)$$

Invertibility:

$$\sum_{j=0}^{K-1} S_{mj} \overline{S}_{jn} = \sum_{j=0}^{K-1} \left(e^{2i\frac{\pi}{K}(m-n)}\right)^j$$

$$= \begin{cases} \dfrac{1-\left(e^{2i\frac{\pi}{K}(m-n)}\right)^K}{1-e^{\frac{2\pi i}{K}(m-n)}} = 0, & m \neq n \end{cases}$$

$L \quad K \quad , \quad m = n$

$\Rightarrow \quad S^1 \cdot S^* = K \cdot I \; , \quad S$ – invertible

(discrete Fourier transform)

$\underline{K \to \infty}$, $\widehat{S^1}_{\text{forms}} \leftrightarrow \widehat{\mathbb{Z}}_{\text{forms}}$ $\quad f(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i x n} f_n$

2) $\boxed{\mathcal{C}_q = \left\{ \underline{\overline{U_q(sl_2)} - \text{mod}} \; / \; \text{projective} \right\}}$ $\quad$ $U_q(sl_2)$ – small quantum sl_2: $K^{2\ell} = 1$, $E^\ell = 0$, $F^\ell = 0$

generated by tensor powers of $V_1 = \mathbb{C}^2$ (indecomp. components of) $\quad \cap \quad \underline{U_q(sl_2)\text{-mod}}$

indecomp. blocks

Ex: $0 \to V_1 \to W \to V_2 \to 0$

$q = e^{\frac{i\pi m}{\ell}}$, $\quad$ irreducible

Ex: V_K, $0 \le K \le \ell - 2$ – irreducibles

Fact: $0 \to V_K \to W_K \to V_{\ell-2-K} \to 0$

is an extension of V_K by $V_{\ell-2-K}$

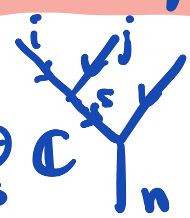
$\mathcal{C}_q$ is a modular tensor category

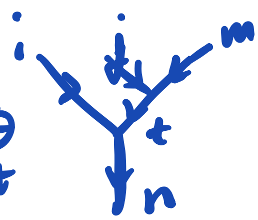
- simple objects $\mathbb{1}, V_1, \dots, V_{\ell-2}$,
- tensor product

$W \cong \bigoplus_i V_i \, W_i$ W_i v.span

$W_i = \mathrm{Hom}(W, V_i)$

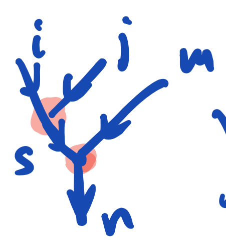
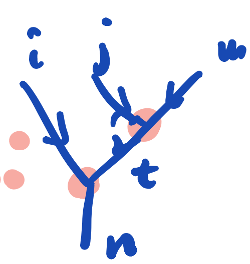
$$V_i \otimes V_j \overset{\text{def}}{=} \bigoplus_{|i-j| \le k \le \min(i+j,\, \ell-2-i-j)} V_k$$

$(V_i \otimes V_j) \otimes V_m = \bigoplus_n \Big(\bigoplus_s \mathbb{C}$  $\Big) V_n$

$V_i \otimes (V_j \otimes V_m) = \bigoplus_n \Big(\bigoplus_t$  $\Big) V_n$

$\mathrm{Hom}((V_i \otimes V_j) \otimes V_m, V_n)$

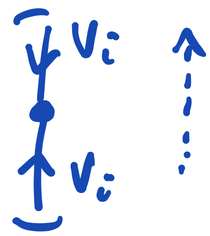
$\mathrm{Hom}(V_i \otimes (V_j \otimes V_m), V_n)$ associator a

$a\Big($  $\Big) = \sum_{t=0}^{\ell-2} \begin{Bmatrix} i & j & s \\ m & n & t \end{Bmatrix}$ 

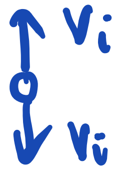
Theorem. when $q = e^{\frac{i\pi}{\ell}}$ q-$6j$ symbols restrict to $0 \le (i, j, \dots) \le \ell-2$

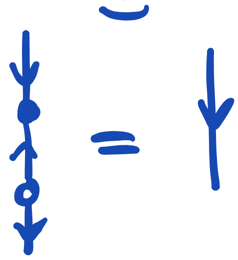
in particular the pentagon identity for generic q gives the pentagon id for $q = \exp\left(\frac{i\pi}{\ell}\right)$.

- $\varphi_i : V_i^* \simeq V_i$, graphically



$\varphi_i^{-1} : V_i \simeq V_i^*$,





- The braiding:

$$V_i \otimes V_j = \bigoplus_{|i-j| \le k \le \min(i+j,\, \ell-2-i-j)} V_k$$

$$V_j \otimes V_i = \bigoplus_{k} V_k$$

(same)

$c_{ij} : V_i \otimes V_j \to V_j \otimes V_i$ is a collection of complex numbers

$$c_{ij}|_{V_k} = (-1)^{i+j-k} \, q^{c_k - c_i - c_j}$$

$c_k = \frac{1}{4} k(k+2)$

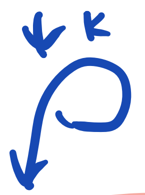
<u>Remark</u> Here, $k = 0, 1, 2, \dots, l-2$;

before, for generic q, used

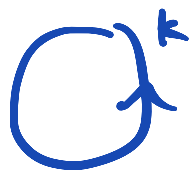
$j = 0, \frac{1}{2}, 1, \dots, \frac{l-2}{2}$ $\qquad j = \frac{k}{2}$

$c_j = j(j+1)$ ← we had it for generic q.

- The ribbon structure

$\mathrm{inv}\Big($  $\Big) = \vartheta_k \cdot \mathrm{id}_{V_k}$, $\quad \vartheta_k = q^{-\frac{k(k+2)}{2}}$

- Functorial dimensions of simple object

$d_k = \mathrm{inv}\Big($  $\Big) = \dfrac{q^{k+1} - q^{-k-1}}{q - q^{-1}} =$

$$= \frac{\sin\left(\frac{\pi}{l} m(k+1)\right)}{\sin\left(\frac{\pi m}{l}\right)}, \quad q = e^{\frac{i\pi m}{l}},$$

when $m = 1$, $d_k > 0$ for all $k = 0, \dots, l-2$

Remark. Same works for $U_q(\mathfrak{g})$ at roots of unity, but there are multiplicities in $V_i \otimes V_j$

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sl_n, $n=3$, multipl. are hard to compute and to work with.

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if invertible

S - matrix
T - ribbon str. matrix

$\left.\begin{matrix} S\text{ - matrix} \\ T\text{ - ribbon str. matrix} \end{matrix}\right\}$ give a representation of $SL_2(\mathbb{Z})$

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for $Sl_n(\mathbb{R})$ princ. series

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\nu N_{\lambda\mu}^\nu V_\nu \, , \quad U_q(\mathfrak{g})$$

$|q|=1$?

$q = e^{i\gamma}$, $\gamma \to 0$, $\lambda = \frac{s}{\gamma}$, $\mu = \frac{t}{\gamma}$, $\nu = \frac{v}{\gamma}$

$s, t, v \in \mathfrak{t}^*$,

Then

$$N_{\lambda\mu}^\nu = \gamma^{-\frac{\dim(\mathcal{M}_{s,t,v}) = d}{2}} \, Vol(\mathcal{M}_{s,t,v}) (1 + O(\gamma))$$

$\mathcal{M}_{s,t,v}$ = the moduli space of G ← compact flat conn. on $S^3 \setminus 3$ pnts with fixed holonomies in conj. classes of $\underbrace{e^{is}, e^{it}, e^{iv}}$

$M_{s,t,v}$ - sympl. (A, B), ω - sympl form

ω^d - volume form.

Y. Taylor, C. Woodward, (for SL_2)

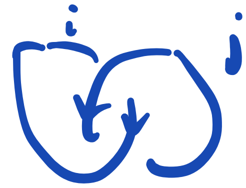
This happens only for $\underline{\underline{m=1}}$.

<u>Remark</u>. The corresp. TQFT and the corresp. CFT are unitary and in particular, conjecherally (in math meaning of this word) its asympl. expansion as $k \to \infty$ coinsides with Witten's quantization of Chern-Simons.

For $m=2$ already, this is not the case... Chen-Yang, ...

$\ell \to \infty$ hyperbolic volumes.

- The S-matrix.

$S_{ij} = \mathrm{inv}\Big($  $\Big) = \sin\left(\frac{\pi}{\ell}(i+1)(j+1)\right)$

Proposition (h.w.)

$$\sum_{k=0}^{\ell-2} \sin\left(\frac{\pi m}{\ell}(x+1)(k+1)\right)\sin\left(\frac{\pi m}{\ell}(y+1)(k+1)\right)$$

$$= \frac{\ell}{2}\,\delta_{x,y}\,,$$

Proof. (h.w.) 1st do it for $m=1$.

In other words

$$S^2 = \frac{\ell}{2}\cdot I\,,$$

— end of examples —

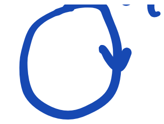
Back the general case of MTC

- Let $N_{ij}^{k} = \dim_{\ell}\big(\mathrm{Hom}(V_i \otimes V_j, V_k)\big)$

mult. of V_k in $V_i \otimes V_j$, can be $N_{ij}^{k} > 1$

- $V_i^* \simeq V_{i^*}$

V_i

- $d_i = \dim_{\mathcal{C}}(V_i) = \mathrm{inv}\Big($  $\Big) = d_{i^*}$

<u>Proposition</u>. 1) $d_i d_j = \sum_k N_{ij}^k d_k$

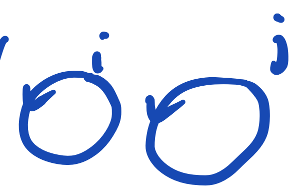
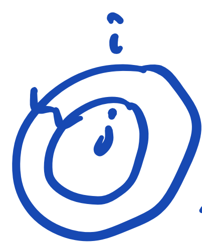
[i.e. $d : [V_i] \to \mathbb{C}$ gives a representation of the commutative Grothendieck ring of $\mathcal{C}$.]

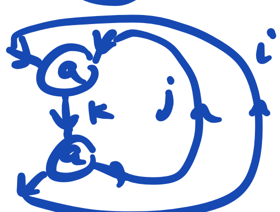
2) $S_{ij} = \sum_k N_{ij^*}^k v_k d_k v_i^{-1} v_j^{-1}$

3) $S_{i^*j^*} = S_{ij}$, $S_{ij} = S_{ji}$

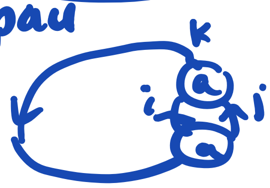
4) $N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^*j^*}^{k^*}$

<u>Proof</u>. (h.w)

1) $d_i d_j = \mathrm{inv}\Big($  $\Big) = \mathrm{inv}\Big($  $\Big)$

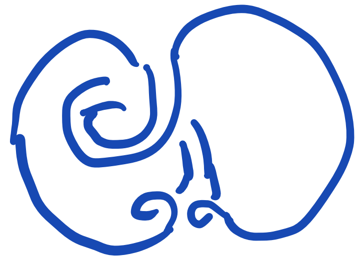
$= \sum_k \sum_a \mathrm{inv}\Big($  $\Big) =$

a basis in the mult. space

$= \sum_k \sum_a \mathrm{inv}\Big($  $\Big) = \sum_k N_{ij}^k \mathrm{inv}\Big($  $\Big)$

$= \sum_k N_{ij}^k \, d_k$,

2) 

$=$  $=$

$= \ldots$ (h/w)