Lect 10.

Last time: $U_q(sl_2)$, $q = e^{\frac{i\pi m}{r}}$, $\ell \leftrightarrow r$

r - odd, m - odd m : r mutually prime.

$\longrightarrow$ an example of Modular Tensor Category $= \overbrace{U_q(sl_2)\text{-mod}}$ / (projective repr.)

generated by $V_1^{\otimes n}$, $V_1 \cong \mathbb{C}^2$

= a semisimple category with finitely many simple objects, can be described "explicitly" in terms of q-6j symboles & ...

Bakalov, Kirillov, ... $U_q(\mathfrak{g})$

H. Andersen & ...; $\rightarrow$ $MTC_q(\mathfrak{g})$, $q^r = -1$,

QFT & TQFT

Functorial aspects.

1. The category of space time.

Typically = a category of cobordisms (1-category)

<u>Ob</u>: $(d-1)$ dim manifolds, with structure (Riemannian, spin, orientation, ...)

<u>Mor</u>: $\mathrm{Hom}(\Sigma_1, \Sigma_2) = \{ d\text{-dim } M$

$\partial M = \overline{\Sigma}_1 \sqcup \Sigma_2 \}$

opposite orientation, we assume all d & $d-1$ dim manifolds in this category are oriented.

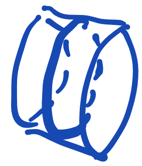
Composition of morphism is the "gluing" M_1, $\partial M_1 = \overline{\Sigma}_1 \sqcup \Sigma_2$

M_2, $\partial M_2 = \overline{\Sigma}_2 \sqcup \Sigma_3$

$M_1 \#_{\Sigma_{2,\varphi}} M_2$ = the comp. of M_1 & M_2

<u>Remark</u>: if M_1, M_2 are smooth,

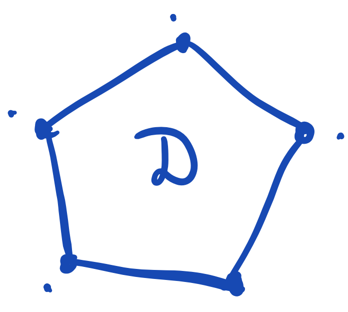
gluing may create corners. More precisely:

Σ should have collars  - S^1 with a 2D collar

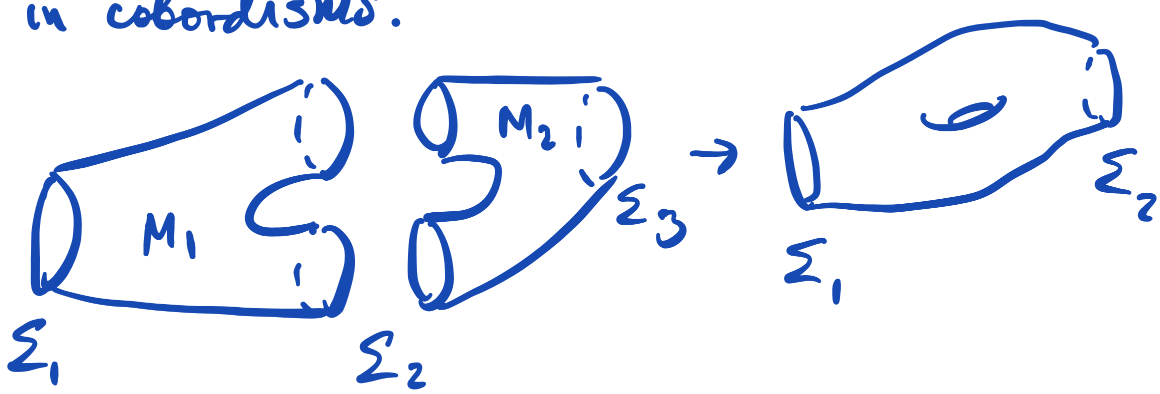
 ← guarantee smoothness.

<u>Remarks</u> 1) "manifolds" ← can be cell complexes

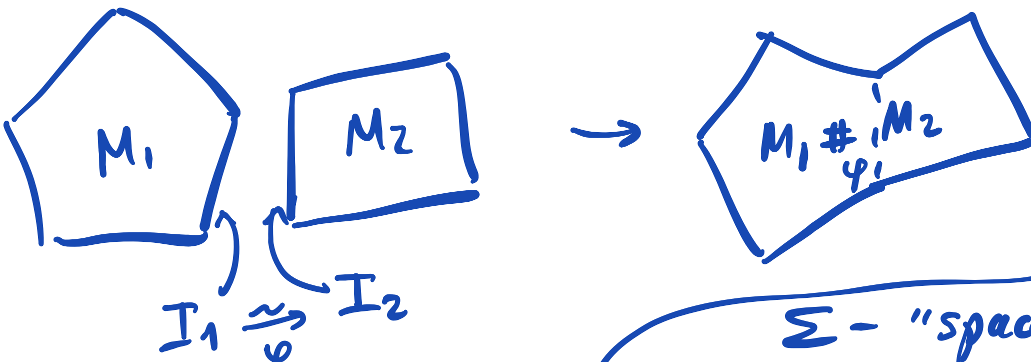
2) can be stratified manifolds



• in cobordisms:



• when M are stratified



2 - categories . . .

Σ – "space" $(d-1)$
M – "space time" (d)

2. Quantum field theory

is a functor

Cobordisms $\longrightarrow$ some "known" category
<u>Vect</u> typically

$$\underset{\text{space}}{\Sigma} \xmapsto{F} \mathcal{H}(\Sigma) \text{ – vector space}$$

(the space of boundary states)

$$M \xmapsto{F} Z(M) : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$$

usually $\mathcal{H}(\Sigma)^* \simeq \mathcal{H}(\bar{\Sigma})$

when Σ, M – oriented, $\mathcal{H}(\Sigma_1)^* \otimes \mathcal{H}(\Sigma_2)$

$$F(M) \in \mathcal{H}(\bar{\Sigma}_1) \otimes \mathcal{H}(\Sigma_2)$$

We assume if $\Sigma = \bigsqcup_\alpha \Sigma_\alpha$

com. comp.

$$\mathcal{H}(\Sigma) = \bigotimes_\alpha \mathcal{H}(\Sigma_\alpha)$$

Then QFT:

- $\Sigma \mapsto \mathcal{H}(\Sigma)$, $\mathcal{H}(\Sigma) = \bigotimes_\alpha \mathcal{H}(\Sigma_\alpha)$

  $\mathcal{H}(\bar{\Sigma}) \simeq \mathcal{H}^*(\Sigma)$

- $M \mapsto Z(M) \in \mathcal{H}(\partial M)$

  the partition func of QFT on M

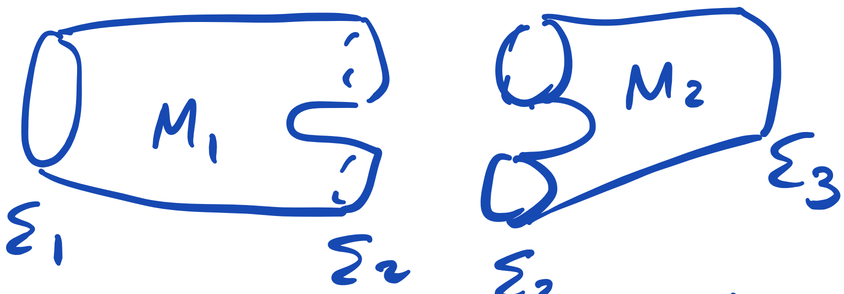
  the space of boundary states

- Comp. law for cobordisms:

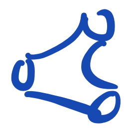
$$Z(M_1 \#_{\Sigma_2} M_2) = Z(M_2) \cdot Z(M_1)$$

$$Z(M_1) : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$$

$$Z(M_2) : \mathcal{H}(\Sigma_2) \to \mathcal{H}(\Sigma_3)$$



$$\begin{tikzcd}
\mathcal{H}(\Sigma_1) \arrow[r, "Z(M_1)"] \arrow[rr, bend right, "Z(M_1 \#_{\Sigma_2} M_2)"'] & \mathcal{H}(\Sigma_2) \arrow[r, "Z(M_2)"] & \mathcal{H}(\Sigma_3)
\end{tikzcd}$$

In this example the Partition fncn ~~ass.~~ for any 2D M is assembled from cylinders & 

<u>locality</u>.

<u>Examples</u>: 1) Quantum mechanics.
(1dim QFT)

- Space time category:

  <u>Ob</u> = pnts

  <u>Mor</u> = $[t_1, t_2] \subset \mathbb{R}$

$$[t_1, t_2] \# [t_2, t_3] = [t_1, t_3]$$

composition.

- The quantum mechanics:

$$F : \text{pnt} \longmapsto \mathcal{H}$$ - the Hilbert space of your system.

$$[t_1, t_2] \longmapsto Z([t_1, t_2]) = e^{\frac{i}{\hbar} \hat{H}(t_2 - t_1)} : \mathcal{H} \circlearrowleft$$

$\hat{H}$ - the Hamiltonian of the quantum system

$\hbar$ - Plank constant

$$Z([t_1, t_2]) Z([t_2, t_3]) = Z([t_1, t_3])$$

(time evolution)

$$e^{\frac{i}{\hbar} \hat{H} s} e^{\frac{i}{\hbar} \hat{H} t} = e^{\frac{i}{\hbar} \hat{H} (t+s)},$$

<u>More generally</u>: $\mathcal{H}(t)$, $Z[t_1, t_2] : \mathcal{H}(t_2) \to \mathcal{H}(t_1)$

For a quantum mech particle in $\mathbb{R}^n$

$$\mathcal{H} = L_2(\mathbb{R}^n), \quad (f,g) = \int_{\mathbb{R}^n} \bar{f} g \, d^n x$$

$$H = -\frac{\hbar^2}{2m}\Delta + V(x)$$

$\uparrow$ an external potential field.

2) An example of ∞-dim QFT

(Secondary quantization) quantum Bose particles, nonrelativistic.

- Space time:

(Space) objects: $\mathbb{R}^n$ (copies of)

Mor (space times) $\mathbb{R}^n \times [t_1, t_2]$,

comp. is the most natural gluing cylinders (as in QM).

The quantum field theory

F : our categ. of space time $\to$ <u>Vect</u>

$F(\mathbb{R}^n)$ = the Fock space $\mathcal{F}$ =

$$= \bigoplus_{N\geq 0} L_2((\mathbb{R}^n)^{\times N})^{S_N}$$

S_N $\uparrow$ Bose

(if Fermi, it would be skew symm.)

is a module over the Heisenberg algebra

- $[\psi(x), \psi(y)] = [\psi^*(x), \psi^*(y)] = 0$, $[\psi(x), \psi^*(y)] = \delta(x-y)$

- Algebra as a vect. space

$$S^\bullet(L_2(\mathbb{R}^n) \oplus L_2(\mathbb{R}^n))$$

ψ $\quad$ ψ^*

$$\psi(f) = \int_{\mathbb{R}^n} f(x)\psi(x)d^n x, \qquad \psi^*(f) = \int_{\mathbb{R}^n} f(x)\psi^*(x)d^n x$$

operators acting on $\mathcal{F}$

- the action: $\mathcal{F} = \bigoplus_{N\geq 0} \left\{ \int_{(\mathbb{R}^n)^{\times N}} f(x_1, \ldots, x_N) \psi^*(x_1)\psi^*(x_2)\ldots\psi^*(x_N)\Omega \right\}$

$$\psi(x)\Omega = 0,$$

This gives $\overbrace{\text{a module str.}}^{\text{Heisenb. alg.}}$ on $\mathcal{F}$.

The space of states is $\mathcal{F}$ and it is a module over Heisenb. algebra.

$$\mathcal{F} = \bigoplus_{N\geq 0} \mathcal{F}_N, \quad \mathcal{F}_0 = \mathbb{C}\Omega$$

$\uparrow$ N-particle subspaces.

The Hamiltonian describing the system of ∞-many particles:

$$\hat{H} = \underbrace{\int_{\mathbb{R}^n} \hbar^2 \frac{\nabla\psi^* \nabla\psi}{2m} d^n x}_{\substack{\text{kinetic term} \\ \text{(free motion)}}} + \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} V(x-y) \cdot \underline{\psi(x)^* \psi^*(y)\psi(y)\psi(x)\, d^n x\, d^n y}$$

$\uparrow$

pairwise interaction with the potential $V(x-y)$.

Claim (h/w), $\hat{H}$ preserves $\mathcal{F}_N$

$$\hat{H}\Big|_{\mathcal{F}_N} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i + \frac{1}{2}\sum_{i\neq j} V(x_i - x_j)$$

The functorial picture:

$$\underset{\text{in time}}{\text{pnts}}\,(\mathbb{R}^n) \mapsto \mathcal{F}$$

$$\mathbb{R}^n \times [t_1, t_2] \longmapsto e^{\frac{i}{\hbar}(t_2 - t_1)\hat{H}} \text{ acting in } \mathcal{F}$$

(nonrelativistic evolution in time).

Path integral:

can "describe" this theory in terms of quasimathematics.

$$\mathcal{F} = \{\text{functional in } \varphi^*: \mathbb{R}^n \to \mathbb{C}\}$$

$$= \{ a + \int_{\mathbb{R}^n} f(x)\varphi^*(x)\,dx + \dots$$

$$\dots + \int f_N(x_1 \dots x_N)\,\varphi^*(x_1)\dots\varphi^*(x_N)\,dx_1$$

$$\int_{(\mathbb{R}^n)^{\times N}} \cdots \qquad \cdots dx_N \}$$

We want to think of $Z([t_1, t_2])$ $: \mathcal{F} \to \mathcal{F}$ as an integral operator with the kernel

$$Z_{[t_1,t_2]}(\varphi^*, \varphi) = $$

$$= \int e^{\frac{i}{\hbar} S[\psi, \psi^*]} \mathcal{D}\psi \, \mathcal{D}\psi^*$$

$$\psi(x, t_1) = \varphi(x)$$

$$\psi^*(x, t_2) = \varphi^*(x)$$

$$\psi, \psi^* : \mathbb{R}^n \times [t_1, t_2] \to \mathbb{C}$$

$$S[\psi, \psi^*] = \int_{t_1}^{t_2} \Big(\int_{\mathbb{R}^n} \Big(-i\hbar \, \psi^* \partial_t \psi + \frac{\hbar^2}{2m} |\nabla \psi|^2 \Big) d^n x - \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} V(x-y) |\psi(x)|^2 |\psi(y)|^2 \, d^n x \, d^n y \Big) dt$$

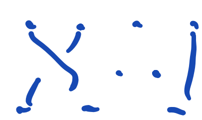
....

Path integral

Meaningful part: $\hbar \to 0$
perturbative, semiclassical.

Our goal: to focus on topological QFT.

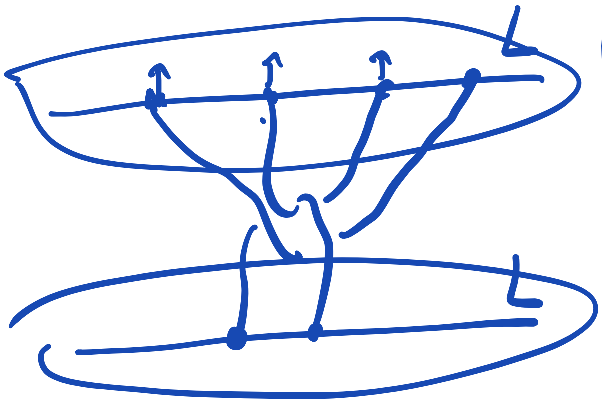
<u>Example</u> (1-dim).
Invariants of tangles.

• space time category 

<u>Ob</u>: <u>enumerated pnts.</u>

<u>Mor</u>: $\underbrace{I \sqcup \cdots \sqcup I}_{k} \sqcup \underbrace{S^1 \sqcup \cdots \sqcup S^1}_{\ell}$

with an extra structure: $\varphi : \sqcup I \sqcup S^1 \hookrightarrow$
$\hookrightarrow \mathbb{R}^2 \times I$, s.t. endpoints $\hookrightarrow L \times$
$\times (\{0\} \sqcup \{1\})$. $L \subset \mathbb{R}^2$ + framing +
+ coloring by $\mathcal{C}$



Composition is gluing,

- <u>A QFT on it</u>:

$$F : \text{spacetimes} \longrightarrow \mathcal{C}$$
$$(\text{framed tangles})$$
$$\mathcal{T}(\mathcal{C}) \longrightarrow \mathcal{C}$$

evaluation functor.

Now we understand that what we did is a 1-dim <u>TQFT</u>

<u>What we want</u>: <u>3d version of this</u>

$$(\text{3-d cobordisms}) \rightarrow (\underline{\text{Vect}})$$

The tool that we will use is the notion of a modular category.

<u>Digression</u>

M - Riemannian, ∂M

$\partial M \longrightarrow$ Functions on the space of boundary fields χ

$$M \mapsto Z_M(\chi) = \int e^{\frac{i}{\hbar} S(\varphi)} \mathcal{D}\varphi$$

$\chi : \partial M \to \mathbb{R}$, $\quad \underline{\varphi|_{\partial M} = \chi}$

$\varphi : M \to \mathbb{R}$, M- Riemannian

$$S(\varphi) = \frac{1}{2} \int_M d\varphi \wedge * d\varphi + \int_M V(\varphi)\, d^n x,$$

M – 3d, A – conn. on a principal G – bundle, locally A is a 1-form

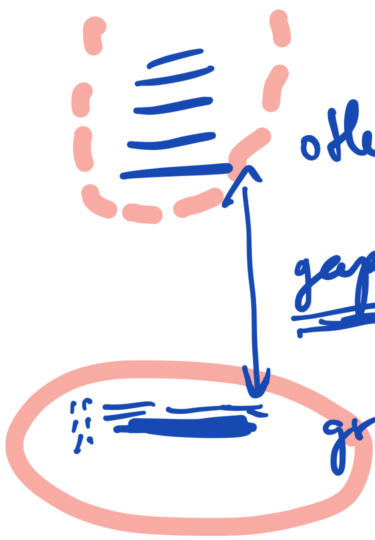
$$S[A] = \int_M \left(\tfrac{1}{2} \underbrace{A \wedge dA}_{3\text{-form}} + \tfrac{1}{3} \underbrace{A \wedge A \wedge A}_{3\text{-form}}\right)$$

Classical topological field theory

$$\int e^{ik S[A]} \, DA$$

Gauge inv.

a model of gauge inv. QFT.

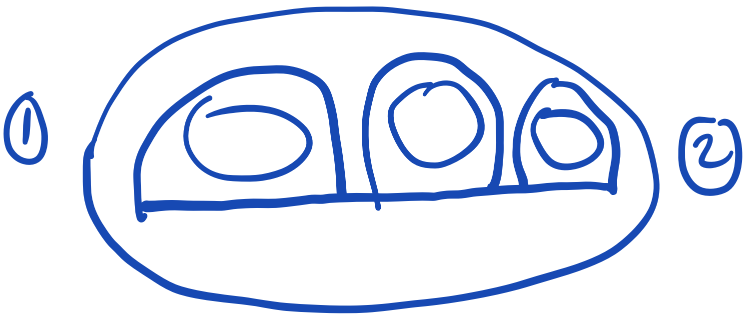


other

gap $\gg$ $E_{char.}$

ground state ← topological sectors of ral QFT.

MTC



M_3 - surgery on a link

(on S_3, on cylinders)