Lect 9

Last time:

- 3D picture for reconsiling tetrahedra associated to q-6j symbols and the world of shadows diagram.

- quantum groups at roots of unity: integral form (over $\mathbb{Z}[q,q^{-1}]$) of $U_q(sl_2)$ with divided powers; specialization to roots of unity.

Digression: other integral forms.

a) it is clear that the subalgebra of $U(sl_2)$, generated by $K^{\pm 1}$, $e=(v-v^{-1})E$, $f=(v-v^{-1})F$ is defined over $\mathbb{Z}[v,v^{-1}]$ with defining relations

$$Ke=v^2eK,\quad Kf=v^{-2}fK,$$

$$[e,f]=(v-v^{-1})(K-K^{-1})$$

with the comultiplication

$$\Delta K = K\otimes K, \quad \Delta e = e\otimes K + 1\otimes e, \quad \Delta f = f\otimes 1 + K^{-1}\otimes f$$

__Note__: when $q=1$ this algebra is commutative but it is noncocommutative Hopf algebra.
It can be identified with the Hopf algebra of polynomial functions over the group

$$SL_2^\vee = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & a^{-1} \end{pmatrix} \right\}$$

by the mapping $\varphi: \mathbb{Z}[K^{\pm 1}, e, f] \to \mathbb{Z}[a^{\pm 1}, c, b]$

$$\varphi(K) = a, \quad \varphi(e) = b, \quad \varphi(f) = c$$

This follows immediately from the multiplication in $SL_2^\vee$:

$$\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & b' \\ 0 & a' \end{pmatrix} = \begin{pmatrix} 1 & ba' + b' \\ 0 & aa' \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ c & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c' & a'^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c + a^{-1}c' & a^{-1}a'^{-1} \end{pmatrix}$$

This is why we will define this integral form $\mathbb{C}_v(SL_2^\vee)$ indicating that it should be regarded as a deformation of functions on $\mathbb{C}(SL_2^\vee)$.

Now let us specialize $v \mapsto q = e^{\frac{i\pi}{\ell}m}$

<u>Thm</u> (i) The subalgebra Z_0 generated by $e^\ell, f^\ell, K^{\pm\ell}$ is central in $\mathbb{C}_q(SL_2^\vee)$

(ii) It is a Hopf subalgebra with
$\Delta K^\ell = K^\ell \otimes K^\ell$, $\Delta e^\ell = e^\ell \otimes K^\ell + 1 \otimes e^\ell$,
$\Delta f^\ell = f^\ell \otimes 1 + K^{-\ell} \otimes f^\ell$

(iii) Small quantum group $u_q(sl_2)$ is the quotient algebra over the central ideal generated by $K^\ell - 1, e^\ell, f^\ell$.

There are more remarkable properties of the algebra $\mathbb{C}_q(Sl_2^\vee)$ (De Concini, Kac).

These Hopf algebras are important to construct invariants of triples $(M, L \subset M, \mathcal{L})$ where M is a 3-manifold, $L \subset M$ is a link and $\mathcal{L}$ is an Sl_2-local system (a module over $\pi_1(M \setminus L)$, or a gauge class of a flat connection on $M \setminus L$ on a principle Sl_2-bundle).

b) One can consider the integral form generated by divided powers in E and by $f = (v - v^{-1}) F$

and by $e = (v - v^{-1}) E$, and by divided powers of F. But since they are not relevant (may be yet) to invariants of tangles and 3-manifolds we will not discuss them here.

Representation theory of $u_q(sl_2)$

Recall that small quantum group $u_q(sl_2)$ is generated by $K^{\pm 1}, E, F$ with defining relations

$$KE = q^2 EK, \quad KF = q^{-2} FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$$K^{2\ell} = 1, \quad E^{\ell} = F^{\ell} = 0$$

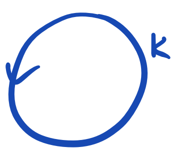
$$\Delta K = K \otimes K, \quad \Delta E = E \otimes K + 1 \otimes E,$$

$$\Delta F = F \otimes 1 + K^{-1} \otimes E,$$

Note that this algebra is not semisimple.

<u>Irreducible representations of $U_q(sl_2)$</u>

$$L_k = V_{\frac{k}{2}}, \quad k = 0, 1, .., \ell - 2$$



$$= tr_{V_k}(K) = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}} = d_k$$

i) $d_k \neq 0$, $k = 0, 1, \ldots, \ell - 2$

ii) $d_{\ell-1} = \dfrac{q^{\ell} - q^{-\ell}}{q - q^{-1}} = 0$

this makes L special.

This makes $L_{\ell-1}$ special

iii) when $q = e^{\frac{i\pi}{\ell}}$, ℓ- odd

$$d_K > 0, \quad K = 0, 1, \dots, \ell-2$$

iv) $$d_{\ell-2} = \frac{q^{\ell-1} - q^{-\ell+1}}{q - q^{-1}} = -\frac{q^{-1} - q}{q - q^{-1}} = 1 = d_0$$

$$d_{\ell-2-K} = \frac{q^{\ell-1-K} - q^{-\ell+1+K}}{q - q^{-1}} = d_K, \quad K = 0, 1, \dots, \ell-2$$

(will be edited)