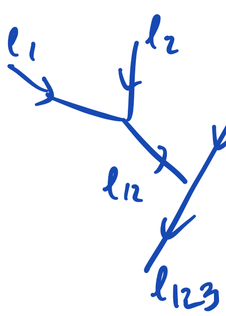
Lecture 8

Last time: - Review of lecture 6.

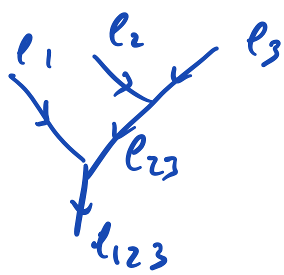
- new formulae for invariants of knots in terms of states of faces.

1) The topology of the world of shadows:

a) The definition of q-6j symbols:

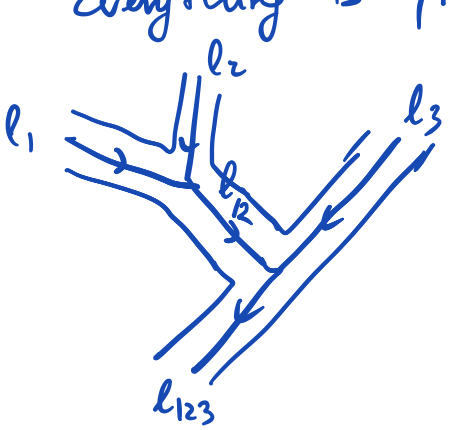


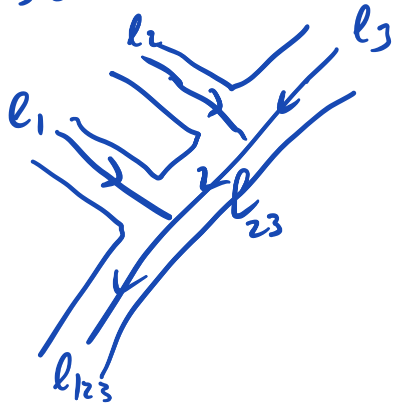
$= \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix}$



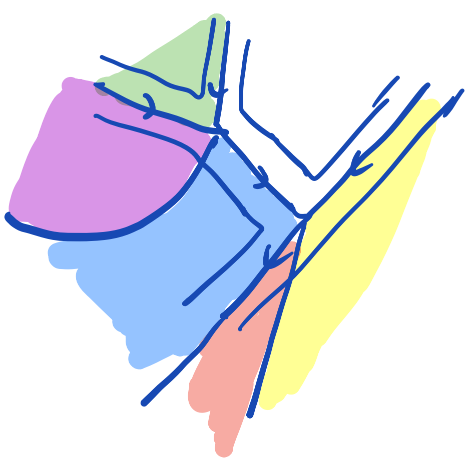
(q6j)

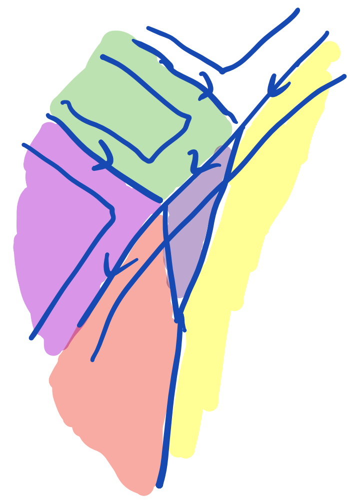
Everything is framed, so



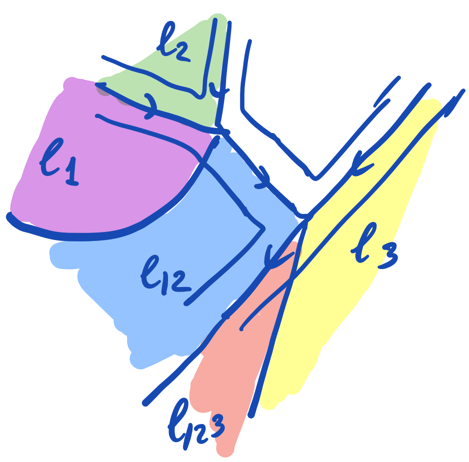


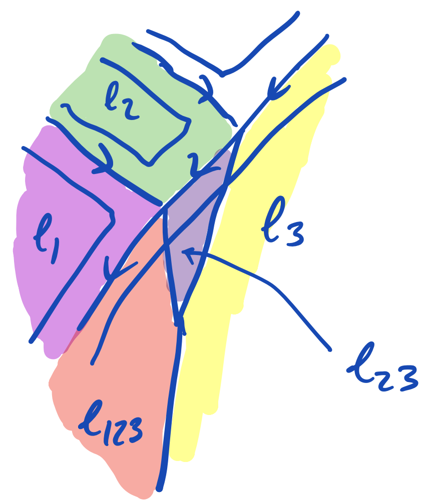
Expand these pictures in 3D :



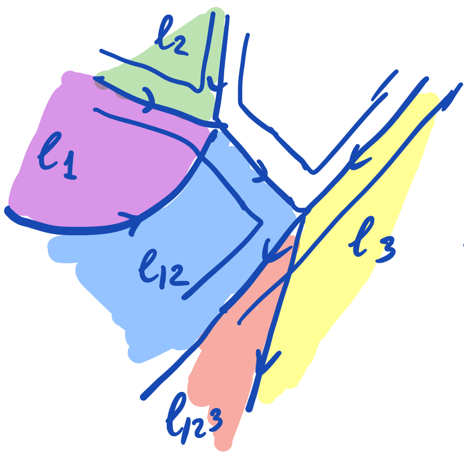


Now "expand" colors down to 3D :

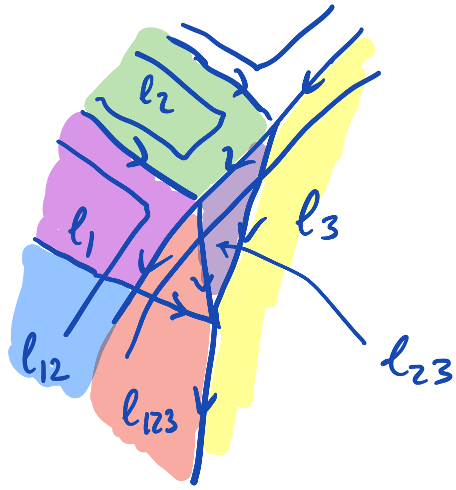




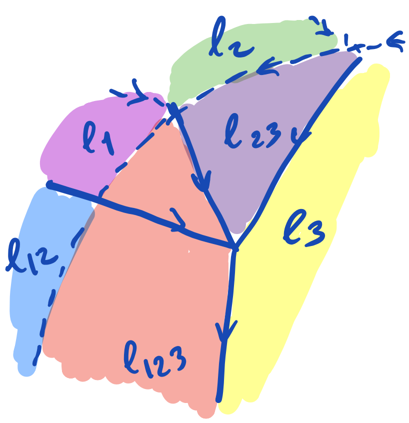
Now identity 96j can be written as



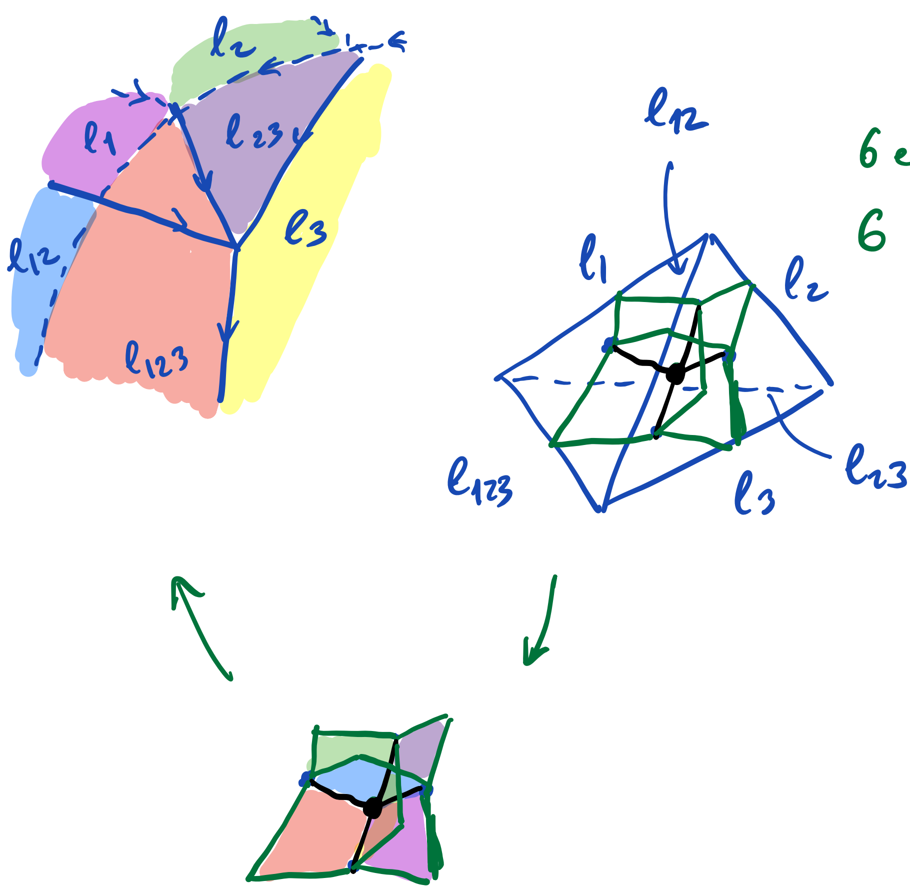
$$= \sum_{\ell_{23}}$$



where we assign to "framed top" q-$3j$ symbols and their compositions and to intersection of 6 planes down in 3D the q-$6j$ symbol:

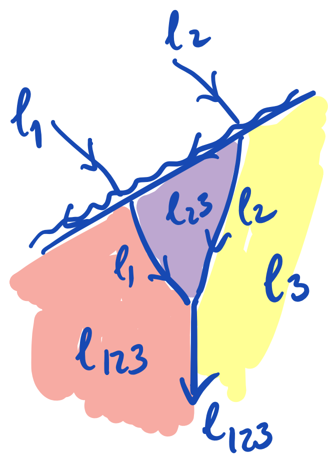


Note that these 6-planes are part of the cell complex which is dual to tetrahedron



6 edges
6 dual planes

The world of shadows that we discussed in lecture 6 is the red-purple-yellow plane:



We will return to this when we will be discussing how RT invariants are connected to Turaev-Viro invariants.

$U_q(sl_2)$ when q is a root of 1

Define the algebra $U_v(sl_2)$ over $\mathbb{Q}(v)$ generated by $E, F, K^{\pm 1}$ with defining relations

$$KE = v^2 EK, \quad KF = v^{-2} FK,$$

$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$

This is a Hopf algebra with

$$\Delta K = K \otimes K, \quad \Delta E = E \otimes K + 1 \otimes E, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F$$

Clearly, we have the homomorphism of Hopf algebras $U_v(sl_2) \to U_h sl_2 [h^{-1}, h]]$

$$K \mapsto e^{\frac{hH}{2}}, \quad v \mapsto e^{h/2}, \quad E \mapsto E, \quad F \mapsto F$$

Note that $R \in U_h sl_2^{\hat{\otimes} 2}$ does not lift to $U_v(sl_2)$.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Denote by $U_{\mathcal{A}}$ the $\mathcal{A}$-subalgebra of $U_v(sl_2)$ generated by 1 and

$$E^{(n)} = \frac{E^n}{[n]!}, \quad F^{(n)} = \frac{F^n}{[n]!}, \quad [n]! = [n][n-1] \cdots [1]$$

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}},$$

Define

$$\begin{bmatrix} K; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K v^{c-s+1} - K^{-1} v^{-c+s-1}}{v^{s} - v^{-s}}, \qquad \begin{bmatrix} K; c \\ 0 \end{bmatrix} = 1$$

$t \in \mathbb{Z}_{\geq 0}$, $c \in \mathbb{Z}$

<u>Thm</u> [L] $U_{\mathcal{A}} \subset U_v(\mathfrak{sl}_2)$ is a Hopf subalgebra.

HW Compute $\Delta E^{(k)}$, $\Delta F^{(k)}$.

<u>Thm</u> [L] Following identities hold

$$E^{(n)} E^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix} E^{(n+m)}, \quad F^{(n)} F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix} F^{(n+m)}$$

$$E^{(p)} F^{(r)} = \sum_{0 \leq t \leq p,r} F^{(r-t)} \begin{bmatrix} K; 2t-p-r \\ t \end{bmatrix} E^{(p-t)}, \quad p, r \in \mathbb{Z}_{\geq 0}$$

$$\begin{bmatrix} K; c \\ t \end{bmatrix} E^{(p)} = E^{(p)} \begin{bmatrix} K; c+2p \\ t \end{bmatrix},$$

$$\begin{bmatrix} K; c \\ t \end{bmatrix} F^{(p)} = F^{(p)} \begin{bmatrix} K; c-2p \\ t \end{bmatrix}$$

<u>Corollary</u> $\begin{bmatrix} K; c \\ t \end{bmatrix} \in U_{\mathcal{A}} \subset U_v(\mathfrak{sl}_2)$

Now we can specialize v to nonzero

complex values. Let $\ell = 1, 3, ..$ be an odd positive integer, mutually prime with an odd $k < \ell$ and $q = e^{i\pi \frac{k}{\ell}}$. We have $q^\ell = -1$.

Now let us specialize v to q. Denote the result by $U_q(sl_2)$.

Proposition Let $m = m_1 \ell + m_0$, $m_0 = 0, .., \ell-1$, $m_1 = 0, 1, 2 \dots$

(a) $$E^{(m_1 \ell + m_0)} = (-1)^{\frac{m_1(m_1-1)}{2} + m_1 m_0} \frac{E^{m_0}}{[m_0]!} \frac{(E^{(\ell)})^{m_1}}{m_1!}$$

and similar identity for F

(b) $U_q(sl_2)$ is generated by $E, F, K^{\pm 1}, E^{(\ell)}, F^{(\ell)}$

Proof. h/w (see Lusztig's book)

Lemma $K^{2\ell} = 1$, $K^\ell \in$ center

Proof. . $K^\ell \in$ center is obviouse

. $$\prod_{s=1}^{\ell} (K v^{-s+1} - K^{-1} v^{s-1}) = \prod_{s=1}^{\ell} (v^s - v^{-s}) \begin{bmatrix} K; 0 \\ \ell \end{bmatrix}$$

$\begin{bmatrix} K;\,0 \\ \ell \end{bmatrix}$ specializes to an element in $U_q(sl_2)$

$$\prod_{s=1}^{\ell}(q^s - q^{-s}) = 0 \Rightarrow \prod_{s=1}^{\ell}\left(Kq^{-s+1} - K^{-1}q^{s-1}\right) = 0 \Rightarrow K^{2\ell} = 1$$

$\blacksquare$

Remark

$$E^{(\ell)}F^{(\ell)} = \begin{bmatrix} K;\,0 \\ \ell \end{bmatrix} + F^{(\ell)}E^{(\ell)} + \sum_{1 \le t \le \ell-1} F^{(\ell-t)} \begin{bmatrix} K;\,2t-2\ell \\ t \end{bmatrix} E^{(\ell-t)}$$

Proposition Elements E, F, $K^{\pm 1}$ generate a finite dimensional Hopf subalgebra $u_q(sl_2) \subset U_q(sl_2)$ with defining relations $K^{2\ell} = 1$, $E^{\ell} = F^{\ell} = 0$ and the usual.

Proof h/w

" $U_q(sl_2) / u_q(sl_2) \simeq U sl_2$ "

(to be edited)