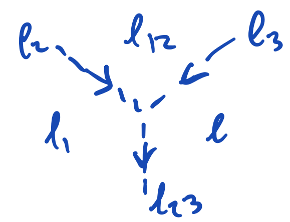
Lect - 7

Last time: – q-$6j$ symbols
– "the world of shadows" or putting states on faces

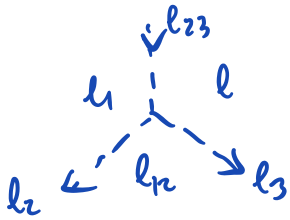
1. Review of lecture 6.

2. Invariants of framed links colored by $U_q(\mathfrak{g})$-modules in the "states on faces" model

Recall that we arrived to the following graphical calculus:

1) 

$\longrightarrow \begin{Bmatrix} l_3 & l_2 & l_{23} \\ l_1 & l & l_{12} \end{Bmatrix}$

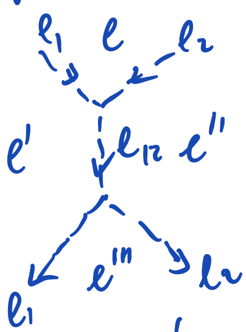


$\longrightarrow \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix}$

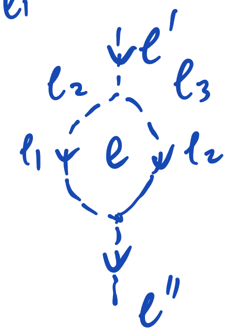
Remark Taking into account the orientation of $\mathbb{R}^2$ the coloring should statisfy the rule

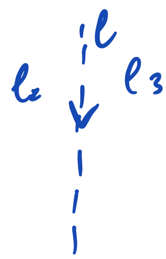
$V_{l_1} \otimes V_{l_2} \supset V_{l_{12}}$

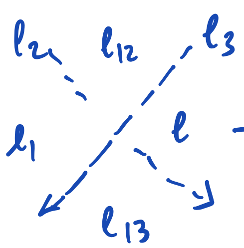
The relations between q-$6j$ symbols now can be written as relations between diagrams in the preimage of the above correspondence:

$\sum_{e_{12}}$ 

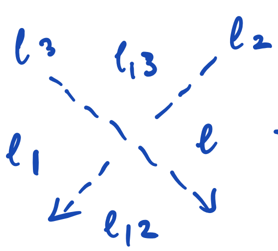
$= \delta_{ee'''}$ 

$\sum_{e}$ 

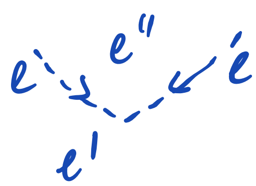
$= \delta_{e'e''}$ 

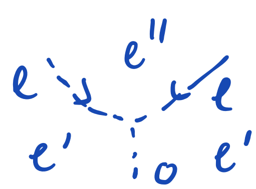
2) 

$\longrightarrow (-1)^{l_{13}+l_{12}-l_1-l}\, q^{\frac{1}{2}(c_l+c_{l_1}-c_{l_{12}}-c_{l_{13}})} \begin{Bmatrix} l_3 & l_1 & l_{13} \\ l_2 & l & l_{12} \end{Bmatrix}$



$\longrightarrow (-1)^{-l_{13}-l_{12}+l_1+l}\, q^{-\frac{1}{2}(c_l+c_{l_1}-c_{l_{12}}-c_{l_{13}})} \begin{Bmatrix} l_2 & l_1 & l_{12} \\ l_3 & l & l_{13} \end{Bmatrix}$

3) 

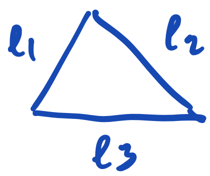
$=$ 

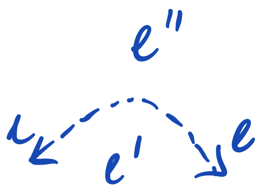
$\longrightarrow \begin{Bmatrix} e & e & 0 \\ e' & e' & e'' \end{Bmatrix}$

$$= (-1)^{l+l''-l'} \left(\frac{[2l'+1]}{[2l''+1]} \right)^{1/2} \delta(l'', l, l')$$

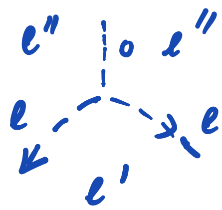
Here $\delta(l_1, l_2, l_3) = \begin{cases} 1, & |l_1 - l_2| \le l_3 \le |l_1 + l_2| \\ 0, & \text{otherwise} \end{cases}$

geometrically $\delta(l_1, l_2, l_3) \neq 0$ if there exists a triangle with side lengths l_1, l_2, l_3





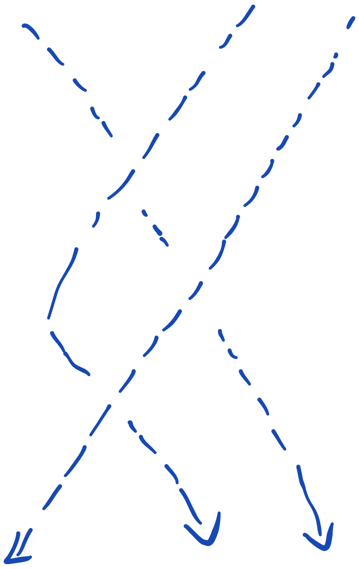
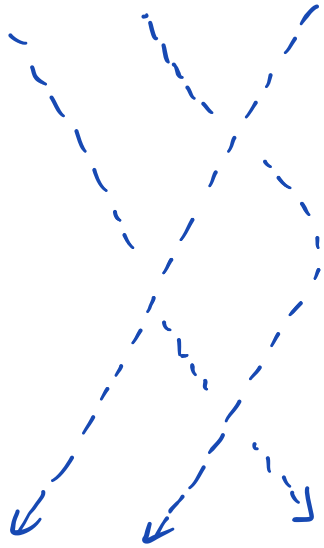
$=$

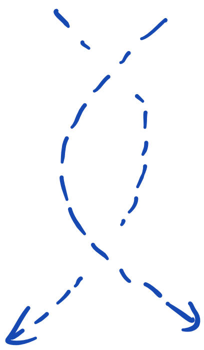
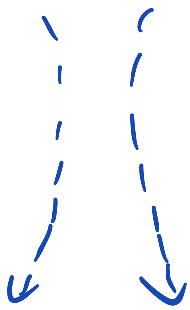


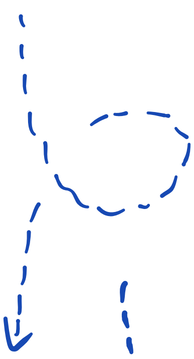
$$\longrightarrow \begin{Bmatrix} l'' & l & l' \\ l & l'' & 0 \end{Bmatrix} =$$

$$= (-1)^{l+l''-l'} \left(\frac{[2l'+1]}{[2l''+1]} \right)^{1/2} \delta(l, l', l'')$$

Redemeister moves are preimages of corresponding relations between q-$6j$ symbols

 = 

 = 

 =  =  v_ℓ, $v_\ell = q^{-c_\ell}$

 =  =  v_ℓ^{-1},

This gives a state sum formula for $U_q(sl_2)$-invariants of links in terms of q-$6j$ symbols.

(to be edited)