Lect 6

Last time: - quantum Weyl group

- refined graphical calculus:

Weyl element and the ½-twist

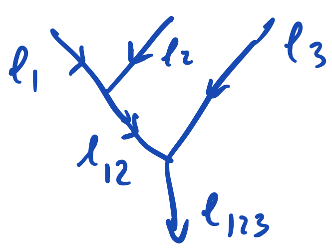
$V_\ell^* \simeq V_\ell$ and change of orientation

- q-$6j$ symbols, tetrahedra.

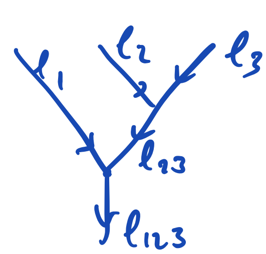
1. $q6j$-symbols and tetrahedra

Recall the basics of graphical calculus for $U_q(sl_2)$

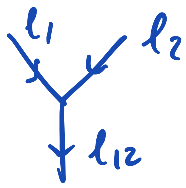
Two bases in $V_{\ell_1} \otimes V_{\ell_2} \otimes V_{\ell_3}$ are related by transition matrix. Its elements are q-$6j$ symbols



$$= \sum_{\ell_{23}} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell_{123} & \ell_{23} \end{Bmatrix}$$



Recall that trivalent vertices here represent embeddings:



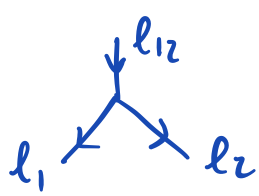
$\rightsquigarrow \quad V_{l_{12}} \to V_{l_1} \otimes V_{l_2}$

in the orthogonal decomposition

$$V_{l_1} \otimes V_{l_2} = \bigoplus_{|l_1 - l_2| \leq l_{12} \leq l_1 + l_2} V_{l_{12}}$$

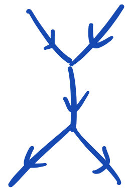
In other words such vertex is a basis element in $\mathrm{Hom}(V_{l_{12}}, V_{l_1} \otimes V_{l_2})$.

Vertex with the opposite orientation represents a basis in $\mathrm{Hom}(V_{l_1} \otimes V_{l_2}, V_{l_{12}})$, i.e. a projection

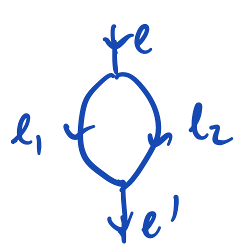


$\longrightarrow \quad V_{l_1} \otimes V_{l_2} \to V_{l_{12}}$

We choose this basis such that

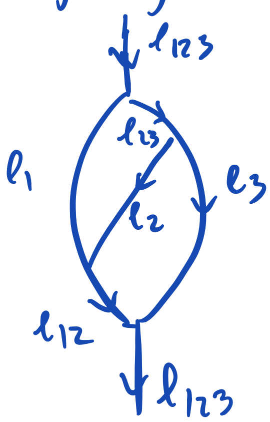


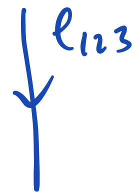
is an orthonormal projector, i.e.



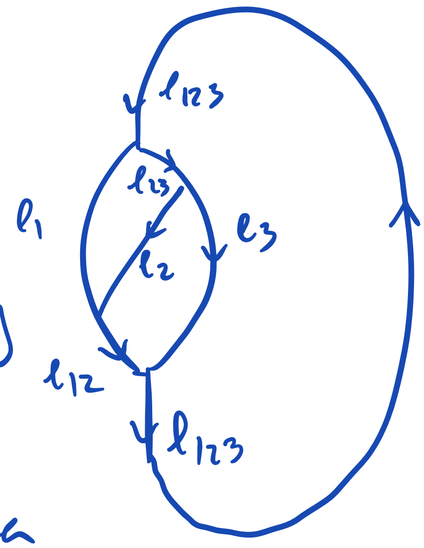
$= \delta_{ee'}$

Using this property we obtain for the q-6j symbol:

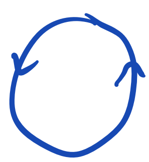
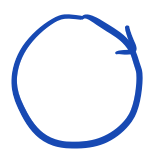
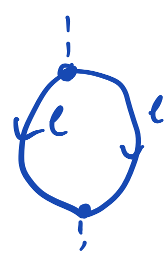


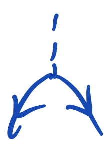
$= \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix}$


or

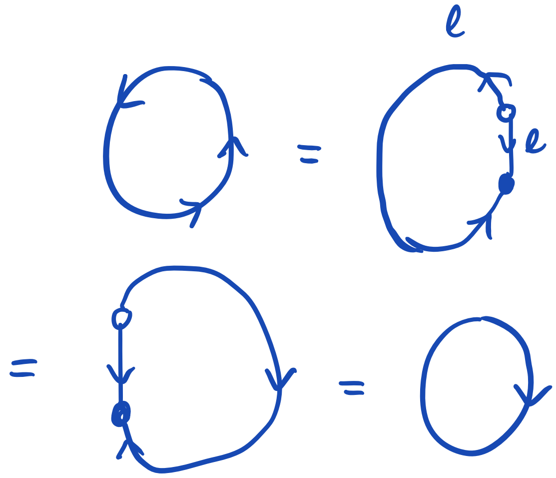
$$\begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix} = \frac{1}{[2l_{123}+1]}$$


Here we used the formula for the quantum dimension of V_l:

$[2l+1] =$  $=$  $= [2l+1]$ 

Here  is the projection to 1-dimensional representation $V_l \otimes V_l \to \mathbb{C}$ and  is the embedding $\mathbb{C} \subset V_l \otimes V_l$.

Note that

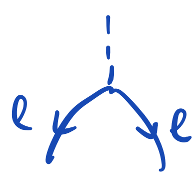


,

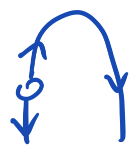
$\sim \varphi : V_\ell^* \to V_\ell$

$\sim \varphi^{-1} : V_\ell \to V_\ell^*$

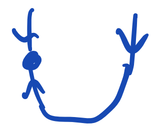
and also



$= \dfrac{1}{\sqrt{[2\ell+1]}}$

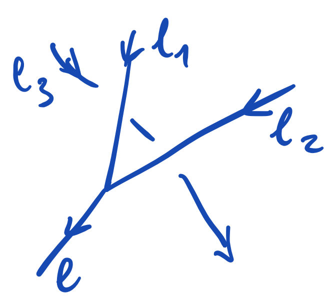


$= \dfrac{1}{\sqrt{[2\ell+1]}}$



Recall some identities for 3j-symbols

(a)



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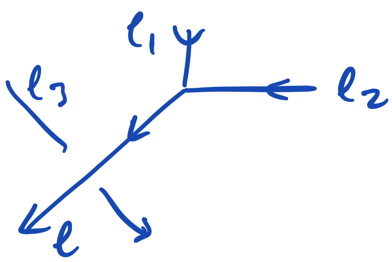
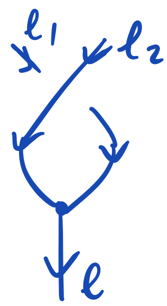
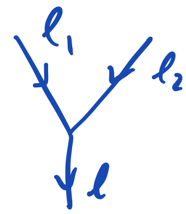
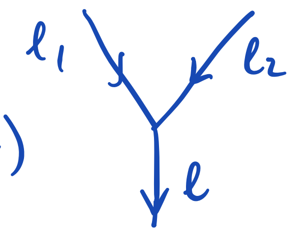
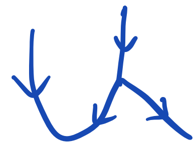


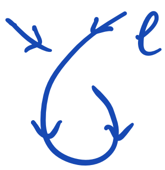
(b) 

$= (-1)^{l_1+l_2-l}\, q^{\frac{1}{2}(c_l - c_{l_1} - c_{l_2})}$ 

(c) 

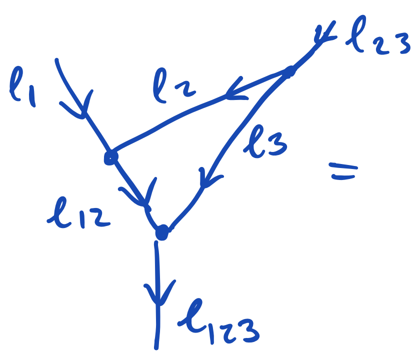
$= (-1)^{l+l_1-l_2} \left(\frac{[2l+1]}{[2l_2+1]} \right)^{1/2}$ 

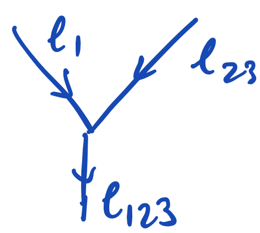
Corollary of (b):



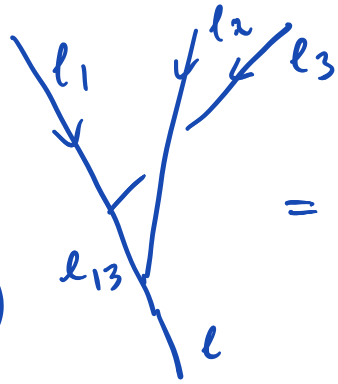
$= (-1)^{2l}\, q^{-c_l}$ 

Now let us return to q-6j symbols.
Orthogonality of 3j symbols implies:

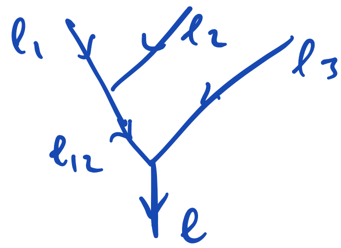
(d) 

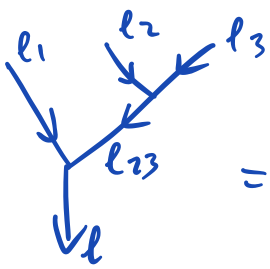
$= \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix}$ 

Thm (*)

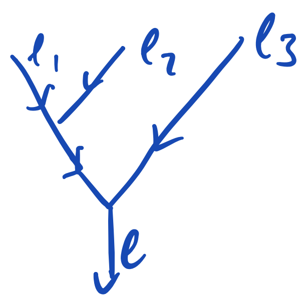
1) 

$$= \sum_{l_{12}} (-1)^{l_{12}+l_{13}-l-l_1} \, q^{\frac{1}{2}(c_l + c_{l_1} - c_{l_{13}} - c_{l_{12}})}$$



2) 

$$= \sum_{l_{12}} \begin{Bmatrix} l_3 & l_2 & l_{23} \\ l_1 & l & l_{12} \end{Bmatrix}$$

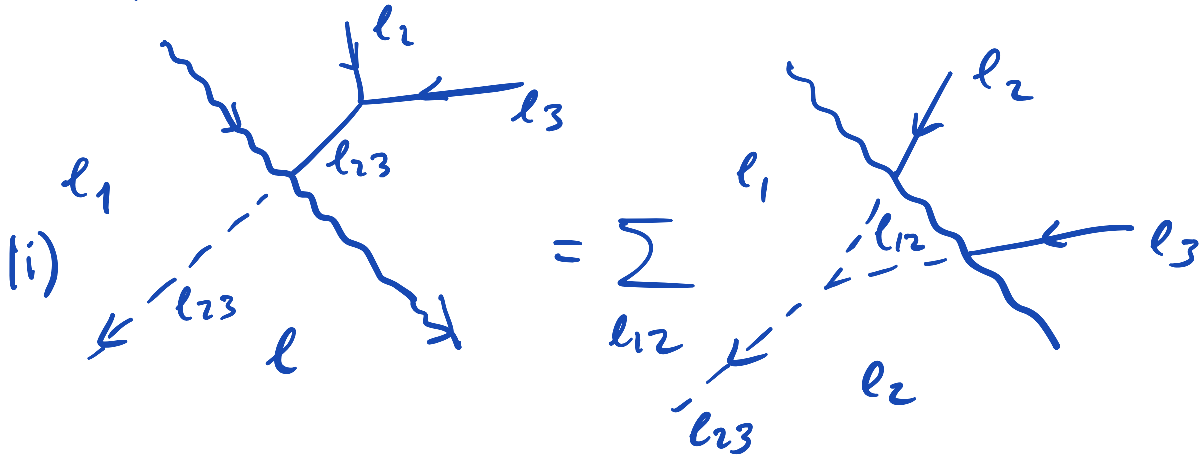


Proof (h/w)

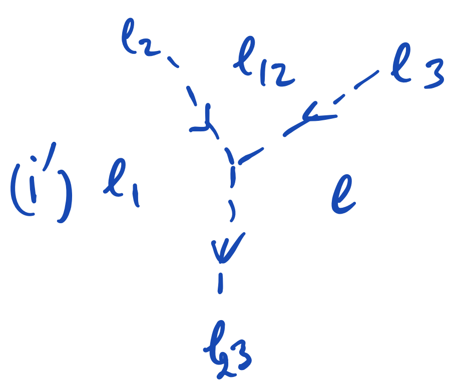
Corollary. Orthogonality of q-6j symbols:

Shadow world for q-6j symbols

Write 2) from Thm $(*)$ as the equality of diagrams

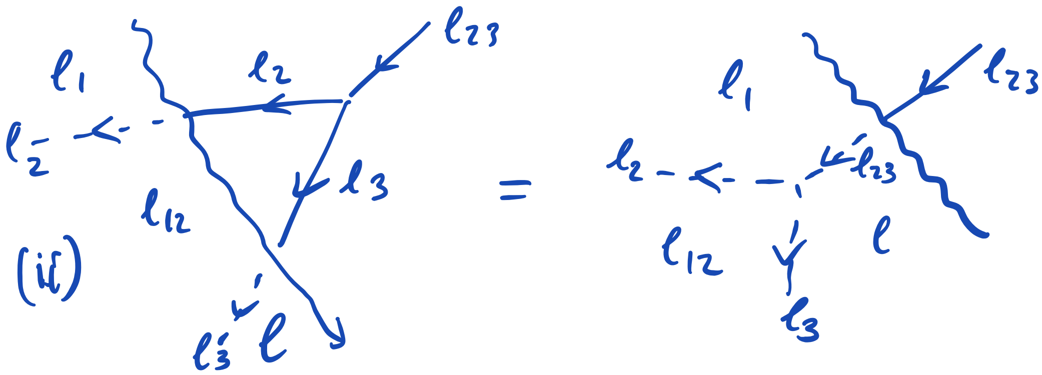


where we assign



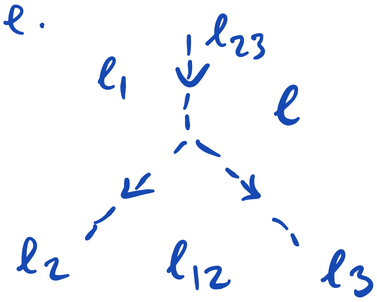
$$\rightsquigarrow \begin{Bmatrix} \ell_3 & \ell_2 & \ell_{23} \\ \ell_1 & \ell & \ell_{12} \end{Bmatrix}$$

Similarly, the identity (α) can be written as



i.e.

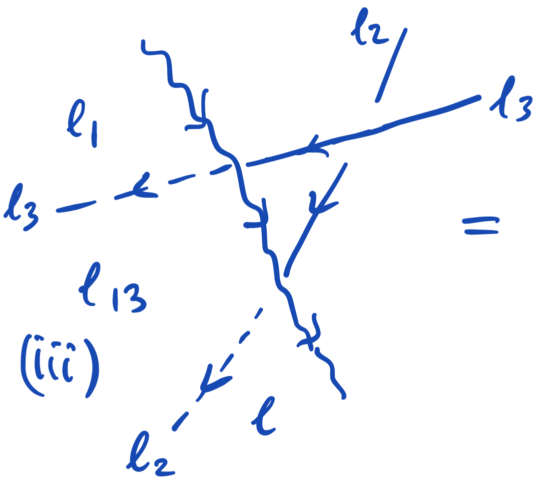
(ii')



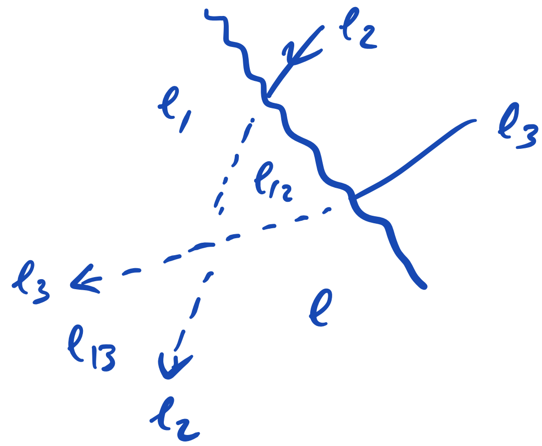
$$\rightsquigarrow \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}$$

The identy (1) in Thm (*) can be written as

(iii)

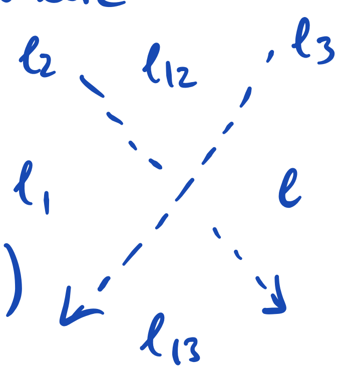


$$= \sum_{\ell_{12}}$$



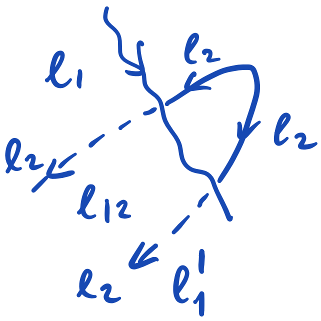
where

(iii')

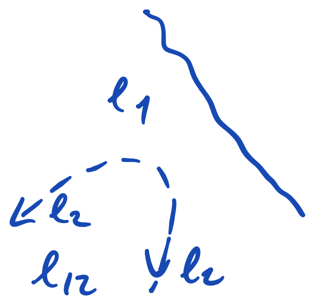


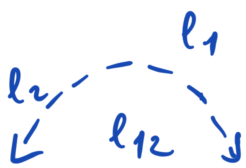
$$\rightsquigarrow (-1)^{\ell_{13}+\ell_{12}-\ell_1-\ell} \, q^{\frac{1}{2}(c_\ell + c_{\ell_1} - c_{\ell_{12}} - c_{\ell_{13}})} \begin{Bmatrix} \ell_3 & \ell_1 & \ell_{13} \\ \ell_2 & \ell & \ell_{12} \end{Bmatrix}$$

A particular case of (ii):



$= \delta_{l_1 l_1'}$

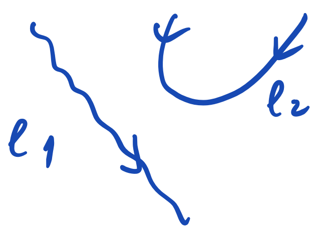




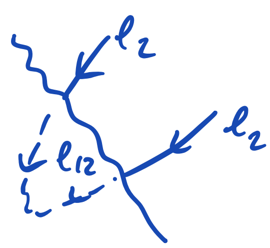
$$\rightsquigarrow (-1)^{l_1+l_2-l_{12}} \left(\frac{[2l_{12}+1]}{[2l_1+1]}\right)^{1/2} \delta(l_1 l_2 l_3)$$

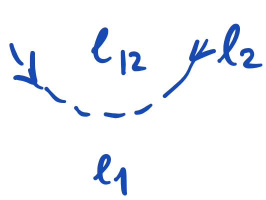
Clebsch-Gordan rules: $|l_1-l_2| \le l_{12} \le l_1+l_2$

Similarly, a particular case of (i):



$= \sum_{l_{12}}$





$$\rightsquigarrow (-1)^{l_1+l_2-l_{12}} \left(\frac{[2l_{12}+1]}{[2l_1+1]}\right)^{1/2} \delta(l_1, l_2, l_3)$$

This gives new formulae for invariants of knots and framed graphs obtained earlier.

(to be edited)