Lect 12

Last time;

- Definition of a modular tensor category (MTC)
- Examples: – twisted category of modules over $\mathbb{Z}_n$

– truncated category of modules over $U_q(sl_2)$, $q = \exp\left(i\frac{\pi m}{\ell}\right)$, m, ℓ - odd and mutually prime.

– Some properties of MTC.

Invariants of closed 3-manifolds

Gluing manifolds along a connected component of the boundary

Let M_1 and M_2 be two manifolds with $\dim(M_1) = \dim(M_2)$ and

$N_1 \subset \partial M_1$, $N_2 \subset \partial M_2$ be connected components of their boundary. Let $f : N_1 \to N_2$ be a homeomorphism.

The manifold

$$M_1 \cup_f M_2 = \{(x,y) \in M_1 \sqcup M_2 \mid y = f(x), \; x \in N_1\}$$

is called the gluing M_1 to M_2 along N_1 and N_2 via f.

<u>Lemma</u> If $\varphi : N_1 \xrightarrow{\sim} N_1$ is a homeomorphism that extends to a homeomorphism $M_1 \xrightarrow{\sim} M_1$ then $M_{f\circ\varphi} \simeq M_f$

If M_1 and M_2 are oriented, N_1 and N_2 are oriented with their orientation induced from M_1 and M_2 and f is an orientation reversing homeomorphism, M_f is naturally oriented.

We will use gluing in construction of 3-manifolds via a surgery and in the construction of a 3-dimensional TQFT.

Homeomorphisms of tori

Isotopy classes of homeomorphisms of a torus $S^1 \times S^1$ form a group $\Gamma_{1,0}$. It is a quotient of the group of all homeomorphisms by the subgroup of isotopies, i.e. homeomorphisms contractible to the identity

Lemma 1 Isotopy classes of homeomorphisms of $S^1 \times S^1$ are in bijection with linear transformations of $H_1(S^1 \times S^1, \mathbb{Z})$, i.e. $\Gamma_{1,0} \simeq SL_2(\mathbb{Z})$.

Let $T_0 = S^1 \times D$ be a standard solid torus. Choose a basis α, β in $H_1(\partial T_0, \mathbb{Z})$ such that $\alpha = [\partial D]$. This fixes an isomorphism $H_1(\partial T_0; \mathbb{Z}) \simeq \mathbb{Z}^2$.

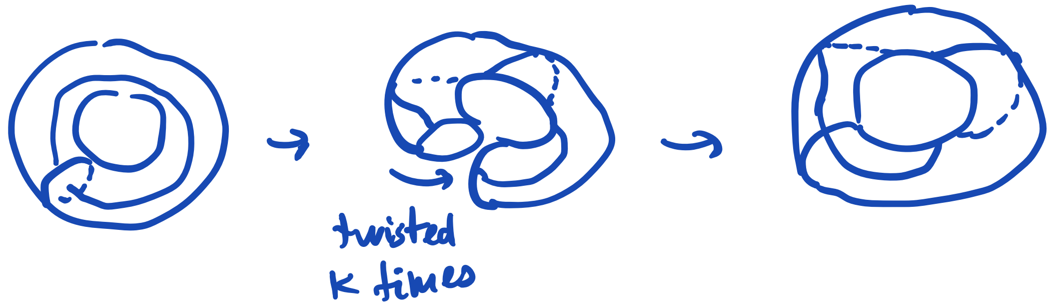
Let f_* be the action of the homeomorphism f on $H_1(\partial T_0; \mathbb{Z})$, $f_* \in SL_2(\mathbb{Z})$. Lemma 1 implies that for any element of $SL_2(\mathbb{Z})$ there exist such homeomorphism, unique up to an isotopy.

<u>Proposition</u>. Homeomorphisms of ∂T_0 representing elements

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}), k \in \mathbb{Z}$$

extend to homeomorphisms of solid torus T_0.

<u>Remark</u>. Homeomorphisms corresponding to $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ are called Dehn twists.



Homeomorphism corresponding to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ changes orientation of ∂T_0.

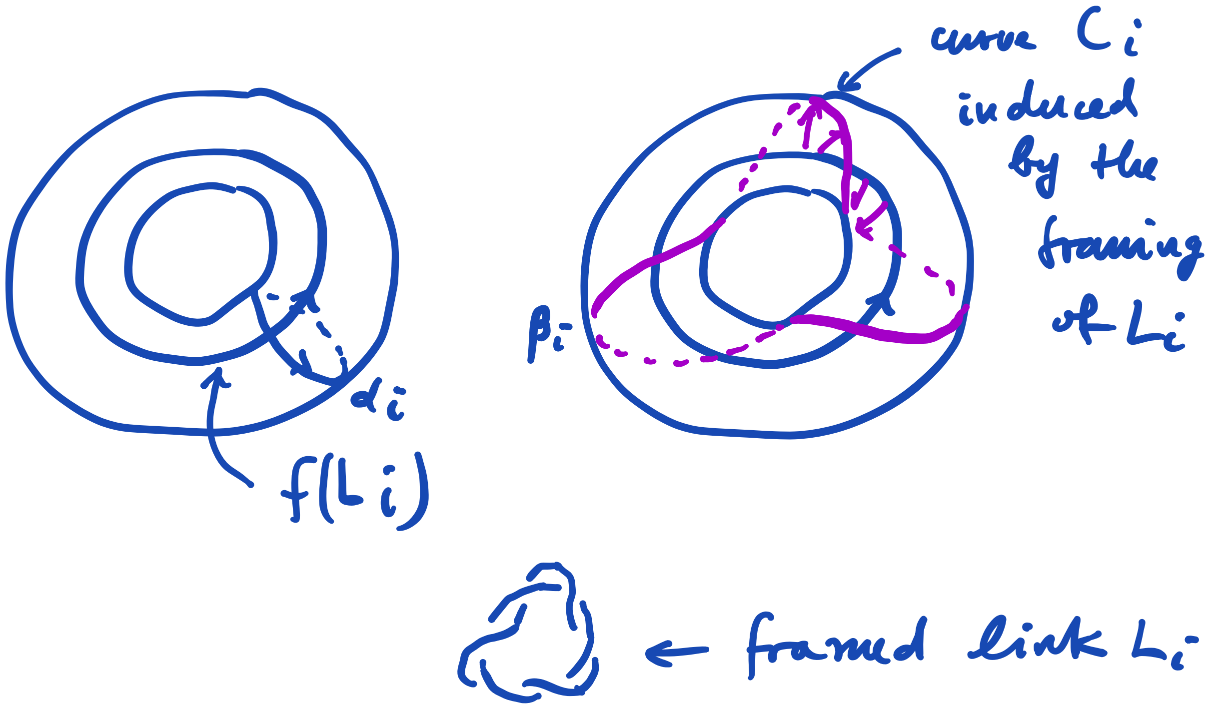
Surgery on S^3

Let $L \subset S^3$ be a framed link with connected components $L_1, \dots, L_m$.

Choose a tubular neighborhood T_i of each connected component L_i.
Each T_i is homeomorphic to $S^1 \times D$ where D is a disc, i.e. 2-dimensional ball B^2. We can choose these

homeomorphisms such that the connected component L_i is mapped to $S^1 \times \{0\}$ where $\{0\}$ is the center of D.

Framing of L_i gives a curve on $\partial(S^1 \times B^2) = S^1 \times \partial D$, $(t, \vec{n}(t))$, $t \in S^1$ where $\vec{n}(t)$ is determined by the framing of L_i, see the picture



We have a natural basis in

$H_1(\partial T_{L_i})$ for each $i=1,\dots,m$: α_i and $\beta_i = [C_i]$.

For each connected component $L_i \subset L$ fix a homeomorphism $f_i : \partial T_i \simeq S^1 \times \partial D$

Now let use these homeomorphisms to glue a copy of $T_0 = S^1 \times D$ to T_i.

Define

$$M_{L,f} = \big(M \setminus \mathrm{Int}(T)\big) \cup_f \left(T_0^{\times m}\right)$$

The 3-manifold $M_{L,f}$ is called the surgery on S^3 along L with attachment maps $f_1,\dots,f_m$.

An orientation of T induces an orientation of the result of the surgery.

Now let us fix isotopy classes of attachment maps f_i. Fix a basis α, β in $H_1(\partial T_0, \mathbb{Z})$ s.t. $\alpha = [\partial D]$ and choose the basis α_i, β_i in $H_1(\partial T_i, \mathbb{Z})$ as above. Let $(f_i)_*$ be the mapping $H_1(\partial T_i, \mathbb{Z}) \to H_1(\partial T_0, \mathbb{Z})$ representing the isotopy class f_i. Choose f_i such that

$$(f_i)_*(\alpha_i) = -\beta, \quad (f_i)_*(\beta_i) = \alpha$$

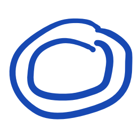
For this choice of attachment maps we will say $M_{L,f} = M_L$ is the result of a standard surgery along L on S^3.

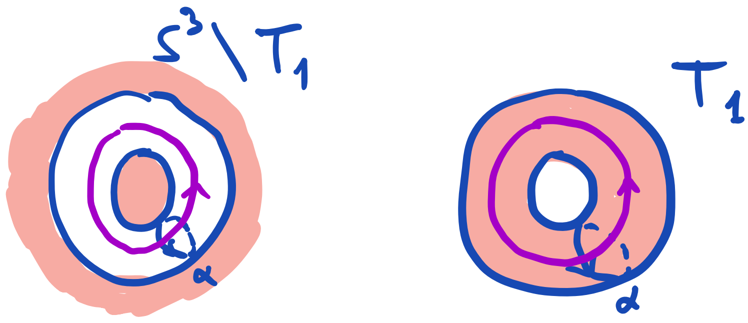
Clearly M_L is a closed, oriented, connected 3-manifold.

Thm. Each closed connected oriented 3-manifold can be obtained as a surgery on S^3 along a framed link.

Remark. M_L does not depend on the orientation of L.

Indeed, change of orientation of $L_i \subset L$ results in a homeomorphism $\alpha \mapsto -\alpha, \beta \mapsto -\beta$ on ∂T_i, which can be extended to a homeomorphism of T_i and therefore M_L is homeomorphic to $M_{\sigma_i(L)}$ where $\sigma_i(L)$ is the link L with opposite orientation of its component L_i.

Examples: (1) $L =$  , unknot with trivial framing (no twists).

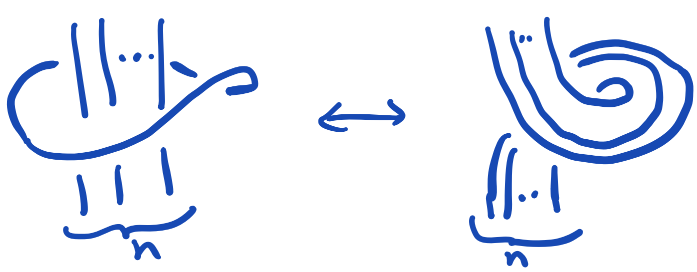


HW. Prove; $M_L = S^2 \times S^1$

(2) $L =$  , unknot with -1 twist of framing

HW. Prove: $M_L = S^3$

<u>Thm</u> (Kirby) $M_L \simeq M_{L'}$ iff L can be obtained from L' be a sequence of moves (Kirby moves):



, $n = 0, 1, 2, \ldots$

More on surgery

Consider B^4 (4-dimensional ball),

$\partial B^4 = S^3$, $L \subset S^3 = \partial B^4 \subset B^4$,

If we choose a tubular neighborhood of L, we have

$$T_L \subset S^3 = \partial B^4 \subset B^4, \quad T_L = \bigsqcup_{i=1}^m T_i$$

where $m = |L|$ is the number of connected components of L.

Now let us glue $\bigsqcup_{i=1}^m (D \times D)$ where D is a disc (two dimensional ball B^2) to B^4 by identifying $(\partial D \times D)_i \xrightarrow[\sim]{g_i} T_{L_i}$.

We assume $T_{L_i} \simeq S^1 \times D$ as above using the framing of L_i.

This gives a compact connected 4-manifold W_L with

$$\partial W_L = (S^3 \setminus \mathrm{Int}(T)) \cup_g (D \times \partial D)^{\times m} = M_L$$

The manifold W_L has natural orientation induced by the orientation of M_L. If $e_1 \wedge e_2 \wedge e_3 \subset T_x M_L$ is an orientation of the tangent space to M_L and $\tau \in T_x W_L$, $x \in \partial W_L$ is the tangent vector such that an open collar in the direction $(-\tau)$ is in W_L (τ is an "outward" tangent vector), then we choose $\tau \wedge e_1 \wedge e_2 \wedge e_3$ as the orientation on $T_x W_L$. It extends to W_L.

Denote by $\sigma(L)$ the signature of the intersection form on $H_2(W_L, \mathbb{R})$. One can show that $\sigma(L)$ is also the signature of the linking matrix of the framed link L.

Invariants of closed 3-manifolds via surgery

Let $\mathcal{C}$ be a modular tensor category.
Recall that if $p_\pm = \sum_{i\in I} d_i \theta_i^{\pm 1}$,
we have

$$p_+ p_- = \sum_{i\in I} d_i^2$$

and the dimension of $\mathcal{C}$ is defined as

$$\mathcal{D} = \sqrt{p_+ p_-}$$

Let $F(L)$ be the invariant of link $L \subset S^3$ colored by the "average" color in $\mathcal{C}$:

$$F(L) = \sum_{i_1 \cdots i_m \in I} d_{i_1} \cdots d_{i_m} \, \mathrm{inv}\left(L_{i_1 \cdots i_m} \right)$$

Here $m = |L|$ is the number of connected components of L and $i_1, \dots, i_m$ is their coloring by simple objects of $\mathcal{C}$. One can fix any orientation of L to construct $\mathrm{inv}(L_{i_1 \dots i_m})$, the invariant $F(L)$ does not depend on it.

Theorem. (i) The combination

$$\tau(M_L) = \mathcal{D}^{-|L|} \left(\frac{p_+}{p_-}\right)^{\frac{\sigma(L)}{2}} F(L)$$

is invariant with respect to the Kirby moves and therefore is an invariant of 3-manifold M_L. Here $\sigma(L)$ is the signature of the linking matrix of the framed link L.

(ii) $\tau(M_{L_1 \sqcup L_2}) = \tau(M_{L_1}) \, \tau(M_{L_2})$

Proof (ii) If $L_1 \subset S^3$ is not linked to $L_2 \subset S^3$, i.e. if they can be separated by encircling L_1 with a ball $B^3 \subset S^3$, $\sigma(L_1 \sqcup L_2) = \sigma(L_1) + \sigma(L_2)$ and

$$F(L_1 \sqcup L_2) = F(L_1) F(L_2)$$

Since $|L_1 \sqcup L_2| = |L_1| + |L_2|$ we proved (ii).

(i) To prove the first statement we have to prove invariance with respect to Kirby moves.

Let us start with $n=0$. Let $L_0 =$ 
We have

$$F(L_0) = \sum_i d_i v_i = p_+$$

and

$$\tau(M_{L_0}) = \mathcal{D}^{-1} \left(\frac{p_+}{p_-}\right)^{\frac{\sigma(L)}{2}} \cdot p_+ =$$

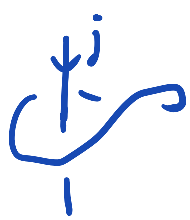
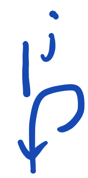
$$= \sqrt{\frac{p_+}{p_-}} \cdot \left(\frac{p_+}{p_-}\right)^{\frac{\sigma(L)}{2}}$$

But for $\sigma(L_0) = -1$ and therefore

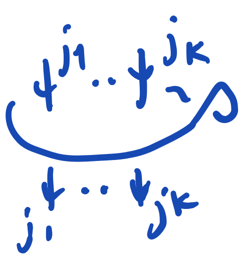
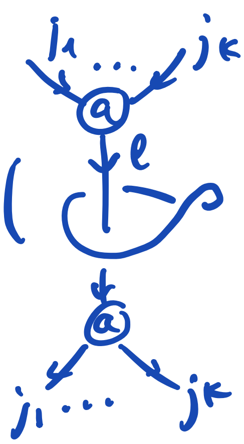
$$\tau(M_{L_0}) = 1$$

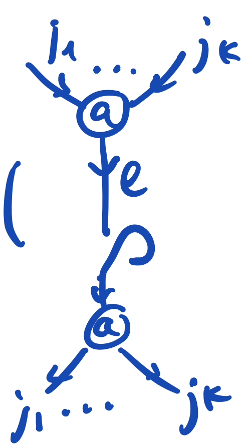
Because $\tau(M_{L \sqcup L_0}) = \tau(M_L)\tau(M_{L_0}) =$ $= \tau(M_L)$ we proved the invariance of τ with respect to $n=0$ Kirby move.

We alredy proved that

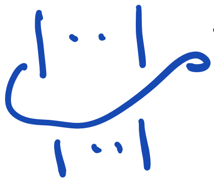
$F\Big($  $\Big) = p_+ F\Big($  $\Big)$

From here:

$F\Big($  $\Big) = \sum_{a,\ell} F\Big($  $\Big) =$

$= \sum_{a,\ell} p_+ F\Big($  $\Big) = p_+ F\Big($  $\Big)$

Thus

$F\Big($  $\Big) = p_+ \, F\Big($  $\Big)$

Denote the link on the left by L_1 and the link on the right by L_2. We have

$$|L_1| = |L_2| + 1, \quad \sigma(L_1) = \sigma(L_2) - 1$$

(h/w, prove this)

Thus

$$\tau(M_{L_1}) = \tau(M_{L_2}) \cdot \mathcal{D}^{-1} p_+ \left(\frac{p_+}{p_-}\right)^{-\frac{1}{2}}$$

$$= \tau(M_{L_2})$$

$\blacksquare$