

## Lect 11

Last time: an outline of the functorial structure of quantum field theory.

### Modular tensor categories

We will consider only semisimple MTC.

- $\mathcal{C}$  is a ribbon category, Abelian
- Finitely many isom. classes of simple objects  $[V_i]_{i \in I}$
- $S_{ij} = \text{inv} \left( \begin{array}{c} V_i \\ \bigcirc \end{array} \begin{array}{c} V_j \\ \bigcirc \end{array} \right)$
- $S$  is invertible

Another important data associated to any MTC is a collection  $\{v_i\}_{i \in I}$ ,  $v_i \in \mathbb{C}^*$  of ribbon elements evaluated on simple objects  $V_i$ :

$$\text{inv}(\downarrow^{V_i}) = r_i \text{ id}_{V_i}$$

We will denote by  $d_i$  the functorial dimension of  $V_i$  :

$$d_i = \text{inv}(\bigcirc^i) = \dim(V_i)$$

Note that  $V_i^*$  is also a simple object. Therefore  $V_i^* \simeq V_{i^*}$  for some  $i^* \in I$ .

This defines an involution

$$*: I \rightarrow I, \quad i \mapsto i^*$$

It is an involution because  $V_i^{**} \simeq V_i$  for ribbon categories.

Examples. 1)  $q = e^{i\frac{\pi}{K}}$ ,  $\mathcal{C}$  has

simple objects  $V_0 = \mathbb{1}, V_1, \dots, V_{K-1}$

- Tensor product :  $V_i \otimes V_j \stackrel{\text{def}}{=} V_{i+j(K)}$ ,
- Braiding :  $C_{ij} : V_i \otimes V_j \rightarrow V_j \otimes V_i$

$$C_{ij} \in \text{Hom}_{\mathbb{C}}(V_{i+j}, V_{i+j}) \simeq \mathbb{C},$$

$$C_{jj'} = \exp\left(\frac{i\pi}{K} j j'\right)$$

- Ribbon structure:  ,  $\sigma_j = e^{\frac{i\pi j^2}{K}}$ ,
- functional dimension  $d_i = 1$
- The S-matrix

$$S_{jj'} = C_{jj'} C_{j'j} = e^{\frac{2\pi i j j'}{K}}$$

Its invertibility

$$\begin{aligned} \sum_{j=0}^{K-1} S_{mj} \bar{S}_{jn} &= \sum_{j=0}^{K-1} (z_m \bar{z}_n^{-1})^j, \quad z_m = e^{\frac{2\pi i m}{K}} \\ &= \begin{cases} \frac{1 - (z_m \bar{z}_n^{-1})^K}{1 - z_m \bar{z}_n^{-1}} = 0, & n \neq m \\ K, & n = m \end{cases} \end{aligned}$$

$$\Rightarrow (S^{-1})_{jn} = \frac{1}{K} e^{-\frac{2\pi i j n}{K}},$$

The S-matrix in this case is the kernel of discrete Fourier transform.

2)  $\widehat{U_q(sl_2)}$ -mod/projective

generated by  $V_1 \cong \mathbb{C}^2$ .

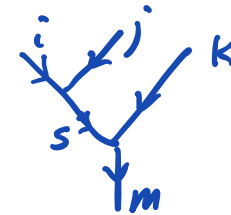
$$q = e^{\frac{i\pi m}{r}},$$

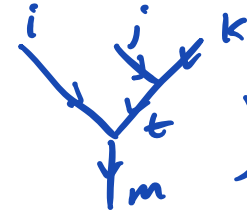
Simple objects: i)  $V_i$ ,  $i=0, 1, \dots, r-2$ ,

ii) Tensor product:

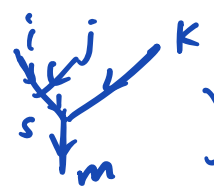
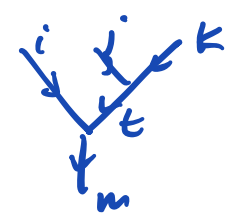
$$V_i \otimes V_j = \bigoplus_{k=|i-j|}^{\min(i+j, r-2-i-j)} V_k,$$

Associativity constraint

$$(V_i \otimes V_j) \otimes V_k = \bigoplus_m \left( \bigoplus_s \text{diagram} \right) \otimes V_m$$


$$V_i \otimes (V_j \otimes V_k) = \bigoplus_m \left( \bigoplus_t \text{diagram} \right) \otimes V_m$$


$$a_{ijk} : (V_i \otimes V_j) \otimes V_k \rightarrow V_i \otimes (V_j \otimes V_k)$$

$$a_{ijk} \left( \text{diagram}_s \right) = \sum_t \left\{ \begin{matrix} i & j & s \\ & k & m & t \end{matrix} \right\} \text{diagram}_t$$



Pentagon identities for the associator follows from such identities for the  $G_j$  symbols.

iii)  $V_i^* \simeq V_i$ ,  $\begin{array}{c} \uparrow i \\ \bullet \\ \downarrow i \end{array}$  and the inverse  $\begin{array}{c} \downarrow i \\ \bullet \\ \uparrow i \end{array}$

$V_0 \simeq \mathbb{1}$  and elements above identify  $\text{Hom}(V_i \otimes V_i, \mathbb{1})$  with  $\text{Hom}(V_i \otimes V_i^*, \mathbb{1})$  and corresponding evaluation maps. The same for injection maps.

iv) Braiding:

$$V_i \otimes V_j = \bigoplus_{k=|i-j|}^{\min(i+j, r-2-i-j)} V_k$$

$$V_j \otimes V_i = \bigoplus_{k=|i-j|}^{\min(i+j, r-2-i-j)} V_k$$

Braiding  $C_{ij}: V_i \otimes V_j \rightarrow V_j \otimes V_i$

is a functorial isomorphism and

therefore acts by automorphisms of  $V_k$ 's  
and we already know

$$c_{ij} |_{V_k} = (-1)^{i+j-k} q^{c_k - c_i - c_j}$$

Here  $c_k = \frac{1}{4} k(k+2)$  and we  
used  $\text{Hom}(V_k, V_k) \simeq \mathbb{C}$

iv) Ribbon structure:

$$\text{inv} \left( \begin{array}{c} \text{↙}^k \\ \text{↘} \end{array} \right) = v_k \text{ id}_{V_k}, \quad v_k = q^{-\frac{k(k+2)}{2}}$$

v) Functorial dimensions (quantum dimensions)

$$d_k = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}} = \frac{\sin \left( \frac{\pi}{r} m (k+1) \right)}{\sin \left( \frac{\pi m}{r} \right)}, \quad q = e^{\frac{2\pi i m}{r}}$$

Note,  $m=1$  is special

$$d_k > 0, \quad k=0, \dots, r-2$$

This is the only value of  $m$  when it is true.

vi) The  $S'$ -matrix:

$$S'_{ij} = \text{inv} \left( \begin{array}{c} i \quad j \\ \text{---} \end{array} \right) = \sin \left( \frac{\pi}{r} (i+1)(j+1) \right)$$

Proposition If  $m, r$  are odd and mutually prime, then

$$\begin{aligned} \sum_{\ell=0}^{r-2} \sin \left( \frac{\pi m}{r} (x+1)(\ell+1) \right) \sin \left( \frac{\pi m}{r} (y+1)(\ell+1) \right) &= \\ &= \frac{r}{2} \delta_{x,y} \end{aligned}$$

Proof. Let us first prove the identity

The sum is

$$\begin{aligned} & \frac{1}{(2i)^2} \sum_{k=0}^{r-2} \left( e^{i \frac{\pi}{r} (k+1)(x+y+2)} - e^{i \frac{\pi}{r} (k+1)(x-y)} + \text{c.c.} \right) \\ &= \frac{1}{(2i)^2} \left( \sum_{k=0}^{r-2} e^{i \frac{\pi}{r} (k+1)(x+y+2)} - \sum_{k=0}^{r-2} e^{i \frac{\pi}{r} (k+1)(x-y)} + \text{c.c.} \right) \end{aligned}$$

If  $\delta_+ = e^{i\frac{\pi}{r}(x+y+z)} \neq 1$ ,  $\delta_- = e^{i\frac{\pi}{r}(x-y)} \neq 1$   
 this expression is equal to

$$\begin{aligned}
 &= \frac{1}{(2i)^2} \left( \frac{1 - e^{i\pi(x+y+z)}}{1 - e^{i\frac{\pi}{r}(x+y+z)}} - \frac{1 - e^{i\pi(x-y)}}{1 - e^{i\frac{\pi}{r}(x-y)}} + \text{c.c.} \right) \\
 &= \frac{1}{(2i)^2} \left( \frac{1-\varepsilon}{1-\delta_+} - \frac{1-\varepsilon}{1-\delta_-} + \frac{1-\varepsilon}{1-\delta_+^{-1}} - \frac{1-\varepsilon}{1-\delta_-^{-1}} \right) = \\
 &= \frac{1-\varepsilon}{(2i)^2} \left( \frac{1}{1-\delta_+} + \frac{\delta_+}{\delta_+-1} - \frac{1}{1-\delta_-} - \frac{\delta_-}{\delta_--1} \right) = \frac{(1-\varepsilon)(1-\varepsilon)}{2i} \\
 &= 0
 \end{aligned}$$

where  $\varepsilon = e^{i\pi(x+y)}$

If  $\delta_- = 1$ , i.e.  $x-y=0$ , and therefore  $\varepsilon=1$

and  $\delta_+ = e^{\frac{2\pi i}{r}x} \neq 1$  since  $x=0,1,\dots,r-2$

it is equal to

$$\frac{1}{(2i)^2} \left( \frac{1-\varepsilon}{1-\delta_+} - r + \frac{1-\varepsilon}{1-\delta_+^{-1}} - r \right) = \frac{r}{2}$$

$\delta_+$  can not be equal to 1 since

if  $x+y+z=2r$  ,  $y=2r-x-z=r, r+1, \dots, 2r-2$   
 when  $x=0, 1, \dots, r-2$  and  $\Rightarrow y$  is not  
 between 0 and  $r-2$ . Thus  $\delta_+ \neq 1$  for  
 all  $x, y=0, 1, \dots, r-2$  □

### Properties of MTC

Let  $\mathcal{C}$  be a modular tensor category.

In our two examples

$$\dim(V_i \otimes V_j, V_k) = 0 \text{ or } 1$$

for all  $i, j, k$ , i.e. the decomposition  
 of the tensor product is multiplicity  
 free. We will use notation

$$N_{ij}^k = \dim(V_i \otimes V_j, V_k)$$

There are important identities involving

$S_{ij}$ ,  $d_i = \dim(V_i)$ ,  $N_{ij}^k$  which we review  
 below.

Recall the definition of  $S_{ij}$ ,

$$S_{ij} = \text{inv} \left( \begin{array}{c} i \quad j \\ \text{Diagram 1} \end{array} \right) =$$

$$= \text{inv} \left( \begin{array}{c} i \quad j \\ \text{Diagram 2} \end{array} \right)$$

Note that if  $\text{MTC}$  is a quotient category of  $\mathcal{U}_q(\mathfrak{sl}_2)$  (or  $\mathcal{U}_q(\mathfrak{g})$ , more generally)

$$S_{ij} = \text{tr}_{V_i \otimes V_j} (R_{21} R_{12})$$

where  $R$  is the universal  $R$ -matrix evaluated on  $V_i \otimes V_j$ .

For any MTC

$$S_{ij} = \text{tr}_{V_i \otimes V_j} (C^{V_j V_i} C^{V_i V_j})$$

where  $\text{tr}_V(f)$  is the functional trace.

Proposition The following identities hold

$$1) \quad N_{ij}^k = N_{ji}^k = N_{i^* k^*}^{j^*} = N_{i^* j^*}^{k^*}$$

$$2) \quad S_{i^* j^*} = S_{ij}, \quad S_{ij} = S_{ji}$$

$$3) \quad d_i d_j = \sum_k N_{ij}^k d_k$$

$$4) \quad S_{ij} = \sum_k N_{ij}^k v_k d_k v_i^{-1} v_j^{-1}$$

Proof 1) Because  $V_i \otimes V_j \cong V_j \otimes V_i$ ,

we have  $\text{Hom}(V_i \otimes V_j, V_k) \cong \text{Hom}(V_j \otimes V_i, V_k)$ .

Dualizing we have

$\text{Hom}(V_i \otimes V_j, V_k) \cong \text{Hom}(V_i \otimes V_k^*, V_j^*)$ , etc.

$$2) \quad S_{ij} = \text{inv} \left( \begin{array}{c} i \\ \curvearrowright \\ j \end{array} \right) = \text{inv} \left( \begin{array}{c} j \\ \curvearrowright \\ i \end{array} \right) \\ = \text{inv} \left( \begin{array}{c} i \\ \curvearrowright \\ i \end{array} \right) = \dots$$

$$3) d_i d_j = \text{inv}(\bigcirc^i) \text{inv}(\bigcirc^j) =$$

$$= \text{inv}(\bigcirc^i \bigcirc^j) = \text{inv}(\bigcirc^{i+j}) =$$

$$= \sum_{k,a} \text{inv} \left( \text{diagram} \right) = \sum_{k,a} \text{inv} \left( \text{diagram} \right) =$$

The first diagram shows a cycle with nodes \$i\$ and \$j\$ and an internal node \$a\$. The second diagram shows a cycle with nodes \$i\$ and \$j\$ and an internal node \$a\$.

$$= N_{ij}^k \text{inv}(\bigcirc^k) = N_{ij}^k d_k$$

4)

$$\text{inv}(\bigcirc^i \bigcirc^j) = \text{inv}(\bigcirc^i \bigcirc^j) =$$

The diagram shows a cycle with nodes \$i\$ and \$j\$ and an internal node \$a\$.

$$= \text{tr}(\sigma_{v_i \otimes v_j}) \sigma_i^T \sigma_j^{-1} = \sum_k N_{ij}^k \sigma_k v_i^T v_j^T.$$



Now define numbers

$$p^{\pm} = \sum_{i \in I} v_i^{\pm 1} d_i,$$

and the "average" color as

$$\text{inv} \left( \downarrow \right) = \sum_{i \in I} d_i \text{inv} \left( \downarrow^i \right)$$

Theorem

$$\text{inv} \left( \begin{array}{c} \text{---} j \\ \text{---} \textcircled{\pm 1} \text{---} \\ \downarrow \end{array} \right) = p^{\pm} \begin{array}{c} \downarrow^j \\ \text{---} \textcircled{\pm 1} \text{---} \\ \downarrow_j \end{array}$$

Here  $\textcircled{\pm 1}$  are left and right twists

$$\begin{array}{c} \downarrow \\ \textcircled{+1} \end{array} = \text{---} \downarrow, \quad \begin{array}{c} \downarrow \\ \textcircled{-1} \end{array} = \text{---} \downarrow$$

Proof

First note that

$$\text{Diagram 1} = \text{Diagram 2} \frac{1}{d_j} \downarrow^j = S_{ij} d_j^{-1} \downarrow^j$$

From this identity

$$\sum_i d_i v_i^{-1} \text{Diagram 1} v_j^{-1} = \sum_i d_i v_i^{-1} S_{ij} v_j^{-1} d_j^{-1} \downarrow^j$$

We also have identities

$$\begin{aligned} \sum_i d_i v_i^{-1} S_{ij} v_j^{-1} d_j^{-1} &= \\ &= \sum_i d_i \text{tr}_{v_i \otimes v_j} (c^{v_j v_i} c^{v_i v_j} v^{-1} \otimes v^{-1}) d_j^{-1} \\ &= \sum_i d_i \text{tr}_{v_i \otimes v_j} (v_{v_i \otimes v_j}^{-1}) d_j^{-1} = \\ &= \sum_{i,k} d_i N_{ij}^k d_k v_k^{-1} d_j^{-1} = \\ &= \sum_{i,k} d_i N_{k^*j}^{i^*} d_k v_k^{-1} d_j^{-1} = \sum_{i,k} d_i N_{kj}^i d_k v_k^{-1} d_j^{-1} \\ &= \sum_k d_k d_j d_k v_k^{-1} d_j^{-1} = p^- \end{aligned}$$

Thus

The diagram shows an equality between two knot diagrams. On the left, a vertical strand with a downward arrow labeled  $j$  passes through a crossing. The crossing is formed by a loop that crosses over the main strand. This loop contains a small circle with a minus sign ( $-1$ ). On the right, the same vertical strand with the downward arrow labeled  $j$  is shown, but without the crossing. An equals sign is placed between the two diagrams.

The proof for  $+$  sign is similar.  $\square$

We will use this theorem to  
construct invariants of 3-manifolds.