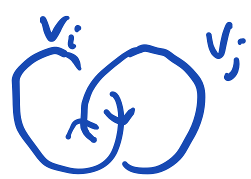
Lect 11

Last time: an outline of the functorial structure of quantum field theory.

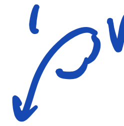
Modular tensor categories

We will consider only semisimple MTC.

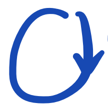
- $\mathcal{C}$ is a ribbon category, Abelian
- Finitely many isom. classes of simple objects $[V_i]_{i \in I}$
- $S_{ij} = \mathrm{inv}\Big($ 

$\Big)$
- S is invertible

Another important data associated to any MTC is a collection $\{v_i\}_{i \in I}$, $v_i \in \mathbb{C}^*$ of ribbon elements evaluated on simple objects V_i :

inv( V_i) $= v_i \, \mathrm{id}_{V_i}$

We will denote by d_i the functorial dimension of V_i :

$d_i =$ inv( i) $= \dim_{\mathcal{C}}(V_i)$

Note that V_i^* is also a simple object. Therefore $V_i^* \simeq V_{i^*}$ for some $i^* \in I$. This defines an involution

$$* : I \to I, \quad i \mapsto i^*$$

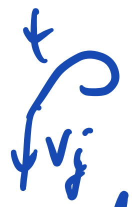
It is an involution because $V_i^{**} \simeq V_i$ for ribbon categories.

<u>Examples</u>. 1) $q = e^{\frac{i\pi}{K}}$, $\mathcal{C}$ has simple objects $V_0 = \mathbb{1}, V_1, \dots, V_{K-1}$

- Tensor product : $V_i \otimes V_j \overset{\text{def}}{=} V_{i+j\,(K)}$,
- Braiding : $c_{ij} : V_i \otimes V_j \to V_j \otimes V_i$

$$c_{ij} \in \mathrm{Hom}_{\mathcal{C}}(V_{i+j'}, V_{i+j}) \simeq \mathbb{C},$$

$$c_{jj'} = \exp\left(\frac{i\pi}{K} jj'\right)$$

- Ribbon structure: 

, $\theta_j := e^{\frac{i\pi j^2}{K}}$,
- functorial dimension $d_i = 1$
- The S-matrix

$$S_{jj'} = c_{jj'}\, c_{j'j} = e^{\frac{2\pi i\, jj'}{K}}$$

Its invertibility

$$\sum_{j=0}^{K-1} S_{mj}\, \bar{S}_{jn} = \sum_{j=0}^{K-1} \left(z_m z_n^{-1}\right)^j, \quad z_m = e^{\frac{2\pi i m}{K}}$$

$$= \begin{cases} \dfrac{1-(z_m z_n^{-1})^K}{1 - z_m z_n^{-1}} = 0, & n \neq m \\ K, & n = m \end{cases}$$

$$\Rightarrow \quad (S^{-1})_{jn} = \frac{1}{K} e^{-\frac{2\pi i\, jn}{K}},$$

The S-matrix in this case is the kernel of discrete Fourier transform.

2) $\overline{\widetilde{U_q(sl_2)}\text{-mod}}/\text{projective}$

$\uparrow$

generated by $V_1 \cong \mathbb{C}^2$. $\qquad q = e^{\frac{i\pi}{r} m}$,

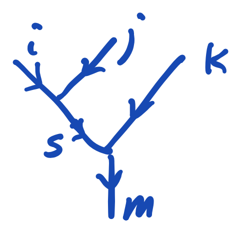
<u>Simple objects</u>: i) V_i , $i = 0, 1, \dots, r-2$,

ii) Tensor product:

$$V_i \otimes V_j = \bigoplus_{k=|i-j|}^{\min(i+j,\, r-2-i-j)} V_k ,$$

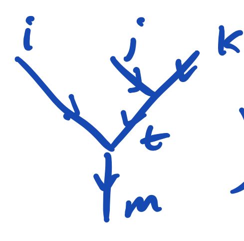
Associativity constraint

$$(V_i \otimes V_j) \otimes V_k = \bigoplus_m \Big(\bigoplus_s$$



$$\Big) \otimes V_m$$

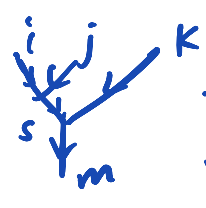
$$V_i \otimes (V_j \otimes V_k) = \bigoplus_m \Big(\bigoplus_t$$



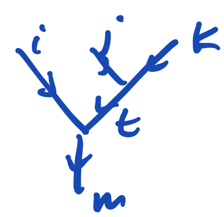
$$\Big) \otimes V_m$$

$$a_{ijk} : (V_i \otimes V_j) \otimes V_k \to V_i \otimes (V_j \otimes V_k)$$

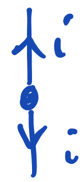
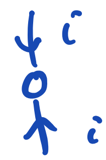
$$a_{ijk} \Big($$



$$\Big) = \sum_t \begin{Bmatrix} i & j & s \\ k & m & t \end{Bmatrix}$$



Pentagon identities for the associator follows from such identities for the 6j symbols.

iii) $V_i^* \simeq V_i$,  and the inverse 

$V_0 \simeq \mathbb{1}$ and elements above identify $\mathrm{Hom}(V_i \otimes V_i, \mathbb{1})$ with $\mathrm{Hom}(V_i \otimes V_i^*, \mathbb{1})$ and corresponding evaluation maps. The same for injection maps.

iv) Braiding:

$$V_i \otimes V_j = \bigoplus_{k=|i-j|}^{\min(i+j,\, r-2-i-j)} V_k$$

$$V_j \otimes V_i = \bigoplus_{k=|i-j|}^{\min(i+j,\, r-2-i-j)} V_k$$

Braiding $c_{ij} : V_i \otimes V_j \to V_j \otimes V_i$

is a functorial isomorphism and

Therefore acts by automorphisms of V_k's and we alredy know

$$c_{ij}\big|_{V_k} = (-1)^{i+j-k}\, q^{c_k - c_i - c_j}$$

Here $c_k = \frac{1}{4}k(k+2)$ and we used $\mathrm{Hom}(V_k, V_k) \simeq \mathbb{C}$

iv) Ribbon structure:

$\mathrm{inv}\Big($  $\Big) = v_k \, \mathrm{id}_{V_k}$, $\quad v_k = q^{-\frac{k(k+2)}{2}}$

v) Functorial dimensions (quantum dimensions)

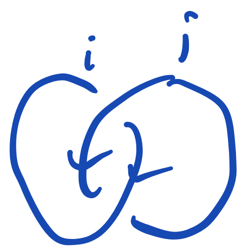
$$d_k = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}} = \frac{\sin\left(\frac{\pi}{r} m (k+1)\right)}{\sin\left(\frac{\pi m}{r}\right)}, \quad q = e^{\frac{i\pi m}{r}}$$

Note, $m=1$ is special

$$d_k > 0, \quad k = 0, \dots, r-2$$

This is the only value of m when it is true.

vi) The S'-matrix:

$S'_{ij} = \mathrm{inv}\Big($  $\Big) = \sin\left(\frac{\pi}{r}(i+1)(j+1)\right)$

<u>Proposition</u> If m, r are odd and mutially prime, then

$$\sum_{\ell=0}^{r-2} \sin\left(\frac{\pi m}{r}(x+1)(\ell+1)\right)\sin\left(\frac{\pi m}{r}(y+1)(\ell+1)\right) =$$

$$= \frac{r}{2}\,\delta_{x,y}$$

<u>Proof.</u> Let us first prove the identity

The sum is

$$\frac{1}{(2i)^2}\sum_{k=0}^{r-2}\left(e^{i\frac{\pi}{r}(k+1)(x+y+2)} - e^{i\frac{\pi}{r}(k+1)(x-y)} + c.c.\right)$$

$$= \frac{1}{(2i)^2}\left(\sum_{k=0}^{r-2} e^{i\frac{\pi}{r}(k+1)(x+y+2)} - \sum_{k=0}^{r-2} e^{i\frac{\pi}{r}(k+1)(x-y)} + c.c.\right)$$

If $\delta_+ = e^{i\frac{\pi}{r}(x+y+2)} \neq 1$, $\delta_- = e^{i\frac{\pi}{r}(x-y)} \neq 1$

this expression is equal to

$$= \frac{1}{(2i)^2}\left(\frac{1-e^{i\pi(x+y+2)}}{1-e^{\frac{i\pi}{r}(x+y+2)}} - \frac{1-e^{i\pi(x-y)}}{1-e^{\frac{i\pi}{r}(x-y)}} + c.c.\right)$$

$$= \frac{1}{(2i)^2}\left(\frac{1-\varepsilon}{1-\delta_+} - \frac{1-\varepsilon}{1-\delta_-} + \frac{1-\varepsilon}{1-\delta_+^{-1}} - \frac{1-\varepsilon}{1-\delta_-^{-1}}\right) =$$

$$= \frac{1-\varepsilon}{(2i)^2}\left(\frac{1}{1-\delta_+} + \frac{\delta_+}{\delta_+-1} - \frac{1}{1-\delta_-} - \frac{\delta_-}{\delta_--1}\right) = \frac{(1-\varepsilon)(1-\varepsilon)}{2i}$$

$= 0$

where $\varepsilon = e^{i\pi(x+y)}$

If $\delta_- = 1$, i.e. $x-y=0$, and therefore $\varepsilon = 1$

and $\delta_+ = e^{\frac{2\pi i}{r}x} \neq 1$ since $x = 0, 1, \dots, r-2$

it is equal to

$$\frac{1}{(2i)^2}\left(\frac{1-\varepsilon}{1-\delta_+} - r + \frac{1-\varepsilon}{1-\delta_+^{-1}} - r\right) = \frac{r}{2}$$

δ_+ can not be equal to 1 since

if $x+y+z=2r$, $y=2r-x-z=r,r+1,\ldots,2r-2$ when $x=0,1,\ldots,r-2$ and $\Rightarrow$ y is not between 0 and $r-2$. Thus $\delta_+ \neq 1$ for all $x,y=0,1,\ldots,r-2$ ∎

Properties of MTC

Let $\mathcal{C}$ be a modular tensor category.

In our two examples

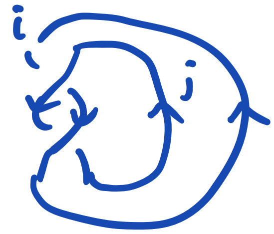
$$\dim(V_i \otimes V_j, V_k) = 0 \text{ or } 1$$

for all i,j,k, i.e. the decomposition of the tensor product is multiplicity free. We will use notation

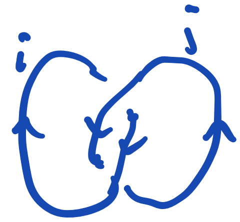
$$N_{ij}^{k} = \dim(V_i \otimes V_j, V_k)$$

There are important identities involving S_{ij}, $d_i = \dim(V_i)$, N_{ij}^{k} which we review below.

Recall the definition of S_{ij},

$S_{ij} = \mathrm{inv}\Big($ 

$\Big) =$

$= \mathrm{inv}\Big($ 

$\Big)$

Note that if MTC is a quotient category of $U_q(sl_2)$ (or $U_q(\mathfrak{g})$, more generally)

$$S_{ij} = \mathrm{tr}_{V_i \otimes V_j}\left(R_{21} R_{12}\right)$$

where R is the universal R-matrix evaluated on $V_i \otimes V_j$.

For any MTC

$$S_{ij} = \mathrm{tr}_{V_i \otimes V_j}\left(c^{V_j V_i} c^{V_i V_j}\right)$$

where $\mathrm{tr}_V(f)$ is the functorial trace.

Proposition The following identities hold

1) $N_{ij}^{k} = N_{ji}^{k} = N_{i\,k^*}^{j^*} = N_{i^* j^*}^{k^*}$

2) $S_{i^* j^*} = S_{ij}$, $S_{ij} = S_{ji}$

3) $d_i d_j = \sum_k N_{ij}^k d_k$

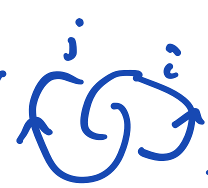
4) $S_{ij} = \sum_k N_{ij}^k v_k d_k v_i^{-1} v_j^{-1}$

Proof 1) Because $V_i \otimes V_j \simeq V_j \otimes V_i$,

we have $\mathrm{Hom}(V_i \otimes V_j, V_k) \simeq \mathrm{Hom}(V_j \otimes V_i, V_k)$.

Dualizing we have

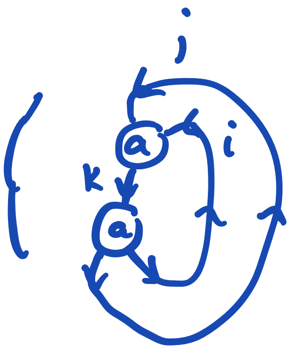
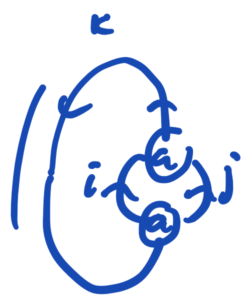
$\mathrm{Hom}(V_i \otimes V_j, V_k) \simeq \mathrm{Hom}(V_i \otimes V_k^*, V_j^*)$, etc.

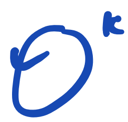
2) $S_{ij} = \mathrm{inv}($  $) = \mathrm{inv}($  $)$

$= \mathrm{inv}($  $) = \ldots$

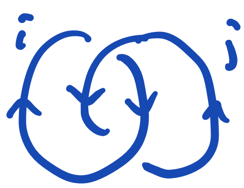
3) $d_i d_j = \mathrm{inv}\Big($  $\Big)\,\mathrm{inv}\Big($  $\Big) =$

$= \mathrm{inv}\Big($  $\Big) = \mathrm{inv}\Big($  $\Big) =$

$= \sum_{k\,a} \mathrm{inv}\Big($  $\Big) = \sum_{k,a} \mathrm{inv}\Big($  $\Big) =$

$= N_{ij}^{k}\ \mathrm{inv}\Big($  $\Big) = N_{ij}^{k} d_k$

4)

$\mathrm{inv}\Big($  $\Big) = \mathrm{inv}\Big($  $\Big) =$

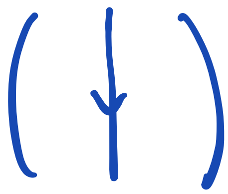
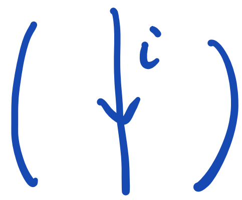
$$= \mathrm{tr}_{V_i \otimes V_j}\left(\nu_{V_i \otimes V_j}\right) \nu_i^{-1} \nu_j^{-1} = \sum_k N_{ij}^{k}\, \nu_k\, \nu_i^{-1} \nu_j^{-1}.$$

∎

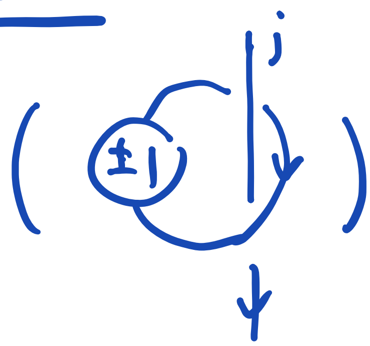
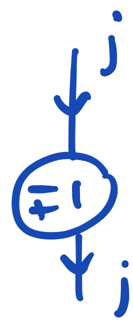
Now define numbers

$$p^{\pm} = \sum_{i \in I} v_i^{\pm 1} d_i \, ,$$

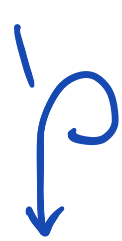
and the "average" color as

inv  $= \sum_{i \in I} d_i$ inv 

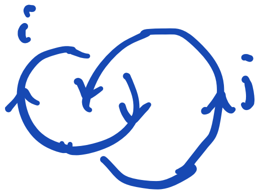
<u>Theorem</u>

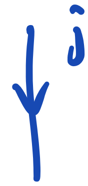
inv  $= p^{\pm}$ 

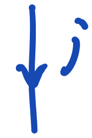
Here  are left and right twists

 =  ,  = 

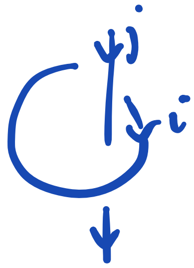
<u>Proof</u> First note that

$=$ 

$\frac{1}{d_j}$ 

$= S_{ij}\, d_j^{-1}$ 

From this identity

$\sum_i d_i v_i^{-1}$ 

$v_j^{-1} = \sum_i d_i v_i^{-1} S_{ij} v_j^{-1} d_j^{-1}$ 

We also have identities

$$\sum_i d_i v_i^{-1} S_{ij} v_j^{-1} d_j^{-1} =$$

$$= \sum_i d_i \,\mathrm{tr}_{V_i \otimes V_j} \left(c^{V_j V_i} c^{V_i V_j} v^{-1} \otimes v^{-1} \right) d_j^{-1}$$

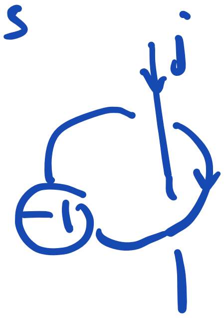
$$= \sum_i d_i \,\mathrm{tr}_{V_i \otimes V_j} \left(v^{-1}_{V_i \otimes V_j} \right) d_j^{-1} =$$

$$= \sum_{ik} d_i N_{ij}^{k} d_k v_k^{-1} d_j^{-1} =$$

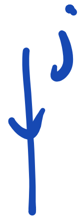
$$= \sum_{ik} d_i N_{k^* j}^{i^*} d_k v_k^{-1} d_j^{-1} = \sum_{ik} d_i N_{kj}^{i} d_k v_k^{-1} d_j^{-1}$$

$$= \sum_k d_k d_j d_k v_k^{-1} d_j^{-1} = p^-$$

Thus



$=\ p^-$



The proof for + sign is similar. ∎

We will use this theorem to construct invariants of 3-manifolds.