Lect 10

Quantum field theory, toplogical quantum field theory

Functorial aspects:

1. The category of space-times. Typically it is a (1-)category of cobordisms

<u>Ob</u>: closed $(d-1)$-dimensional manifolds (possibly with structures such as Riemannian, pseudo-Riemannian, spin, oriented, ...)

<u>Mor</u>: If Σ_1 and Σ_2 are two objects,

$Hom(\Sigma_1, \Sigma_2) = \{$ d-dimensional manifolds M, s.t. $\partial M = \overline{\Sigma_1} \sqcup \Sigma_2$

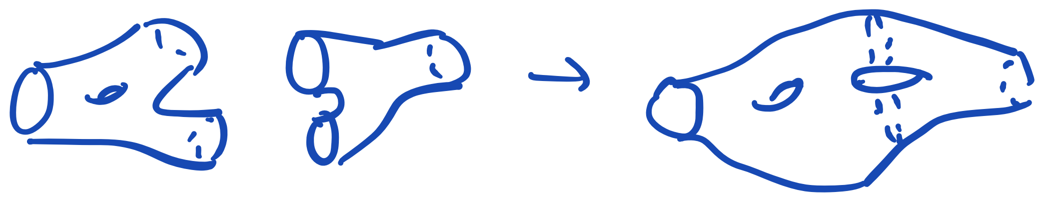
$\overline{\Sigma_1}$ has opposite orientation of Σ_1,

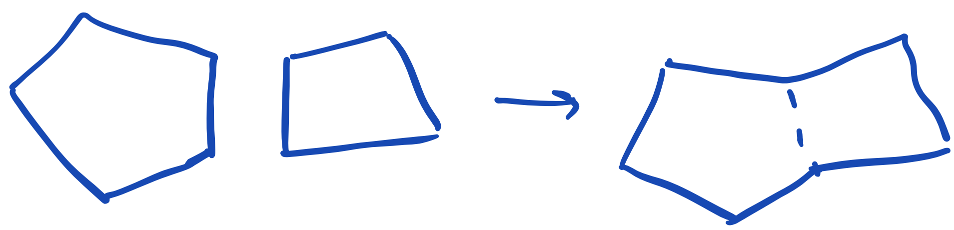
structures on M and $\overline{\Sigma_1} \sqcup \Sigma_2$ agree

(for example metric).

Remarks. 1) manifolds can be cell complexes.

2) Cobordisms can be extended to cell complexes or stratified manifolds with more complicated structure.

a) 

b) 

b) corresponds to higher categories, but we will not discuss this here.

2. Quantum field theory, functorially is a functor

Cobordisms $\to$ Fixed target category, usually <u>Vect</u>

i.e.

$$\underset{(d-1)}{\Sigma} \longmapsto \mathcal{H}(\Sigma) \quad \text{vect. space}$$

$$M \longmapsto Z(M) : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$$

$\partial M = \overline{\Sigma_1} \sqcup \Sigma_2$

Change of orientation: $\mathcal{H}(\overline{\Sigma}) \simeq \mathcal{H}(\Sigma)^*$

(in the appropriate top. sense when $\dim(\mathcal{H}(\Sigma)) = \infty$)

$$M \mapsto Z(M) \in \mathcal{H}(\Sigma_2) \otimes \underset{\underset{\mathcal{H}(\Sigma_1)^*}{\simeq}}{\mathcal{H}(\overline{\Sigma_1})}$$

Covariant functor:

$$Z(M_2) \circ Z(M_1) = Z(M_2 \underset{\Sigma_2}{\#} M_1),$$

$$M_1 : \Sigma_1 \to \Sigma_2, \quad M_2 : \Sigma_2 \to \Sigma_3,$$

If the QFT functor is defined on the category of topological cobordisms we have a topological quantum field theory.

<u>Examples</u>: 1) Invariants of tangles.

1- dimensional cobordisms with extra structure.

<u>Objects</u>: ordered points (oriented, $\pm$)

<u>Morphisms</u>: $M = I^{\times k} \times S^{1 \times e}$ with an isotopy class of embedding to $\mathbb{R}^2 \times I$ s.t. endpoints are on $L \times (\{0\} \cup \{1\}) \subset \mathbb{R}^2 \times I$, $L \subset \mathbb{R}^2$ (special structure)

The TQFT is given by invariants of tangles.

2) Quantum mechanics:

Space time category:

<u>Objects</u>: pnt

<u>Morphisms</u>: Euclidean intervals

QFT (QM) is $F: \text{Spacetime} \to \text{Vect}$

$$F(\text{pnt}) = \mathcal{H} \text{ - Hilbert space}$$

$$F([t_1,t_2]) = e^{\frac{i(t_2-t_1)H}{\hbar}} : \mathcal{H} \to \mathcal{H}$$

Gluing property = the semigroup property of the propagator

$$F([t_2,t_3])\,F([t_1,t_2]) = F([t_1,t_3])$$

In quantum mechanics on $\mathbb{R}^n$

$$\mathcal{H} = L_2(\mathbb{R}^n), \quad H = -\frac{\hbar^2 \Delta}{2m} + V(x)$$

3) A quantum field theory

<u>Space time category</u>: Euclidean cobordisms

$\underline{Ob}$: $\{\mathbb{R}^n\}$

$\underline{Mor}$: $\quad \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) = \{\mathbb{R}^n \times \underset{[t_1,t_2]}{I}\}$

$F : \text{Spacetime} \longrightarrow \underline{\text{Vect}}$

$$F(\mathbb{R}^n) = \mathcal{F} = \bigoplus_{N \geq 0} L_2\left((\mathbb{R}^n)^{\times N}\right)^{S_N}$$

Heisenberg algebra: $[\psi(x), \psi(y)] = [\psi^*(x), \psi^*(y)] = 0$

$$[\psi(x), \psi^*(y)] = \delta(x-y), \qquad x, y \in \mathbb{R}^n$$

$$\left(\psi[f] = \int_{\mathbb{R}^n} \psi(x) f(x) dx, \quad \psi^*[f] = \int \psi^*(x) f(x) dx\right)$$

$$\text{Algebra} = S^{\cdot}\left(L_2(\mathbb{R}^n) \oplus L_2(\mathbb{R}^n)\right)^{\mathbb{R}^n}$$

Acts on $\mathcal{F}$ as:

$$\psi(x)\Omega = 0$$

$$\mathcal{F} = \bigoplus_{N \geq 0} \left\{ \int_{(\mathbb{R}^n)^N} g(x_1, \dots, x_n) \psi^*(x_1) \dots \psi^*(x_n) \Omega \right.$$

$$F(\mathbb{R}^n \times [t_1, t_2]) = \exp\left(\frac{i}{\hbar} \hat{H} (t_2 - t_1)\right) = \hat{U}_{t_2 - t_1}$$

$$\hat{H} = \int_{\mathbb{R}^n} \hbar^2 \frac{\nabla\psi^* \nabla\psi}{2m} \, dx + \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} V(x-y)\, \psi^*(x)\psi^*(y)\psi(y)\psi(x) \, dx\, dy$$

$$\hat{H}\Big|_{\mathcal{F}_N} = -\hbar^2 \sum_{i=1}^{N} \Delta_i + \frac{1}{2}\sum_{i \neq j} V(x_i - x_j)$$

Non-relativistic quantum Bose gas.

<u>Path integral</u>: $\mathcal{F}$ = functionals in $\varphi^*(x)$

$$\hat{U}_{t_2 - t_1}(\varphi^*, \varphi) = \int_{\substack{\psi(x,t_1) = \varphi(x) \\ \psi^*(x,t_2) = \varphi^*(x)}} e^{\frac{i}{\hbar} S(\psi, \psi^*)} \, \mathcal{D}\psi\, \mathcal{D}\psi^*$$

$$S(\psi, \psi^*) = \int_{t_1}^{t_2} \left(\int_{\mathbb{R}^n} \left(-i\hbar\, \psi^* \partial_t \psi + \hbar^2 \frac{\nabla\psi^* \nabla\psi}{2m} \right) - \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} V(x-y)\, |\psi(x)|^2 |\psi(y)|^2 \, dx\, dy \right) dt$$

(will be edited)