Superintegrable Systems. II.

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Last time: superintegrable models classical and quantum.

Classical:

$$\begin{array}{c} \overset{\text{symplectic}}{M_{2n}} \longleftarrow \text{generic fiber} = \\ = \sqcup \text{ isotropic } k\text{-dim submanifolds} \\ \downarrow p_1 \\ P_{2n-k} \longleftarrow \text{generic fiber} = \sqcup \text{ sympl. leaves} \\ \text{Poisson} \\ \downarrow p_2 \\ B_k \quad k<n \\ \nwarrow \text{trivial Poisson} \end{array}$$

$$\begin{array}{c} M_{2n} \leftarrow \text{gen. fibers are Lagrangian } n\text{-dim.} \\ \downarrow \\ B_n \end{array}$$

Algebraically:

$$C(M_{2n}) \hookleftarrow J_{2n-k} = C(P_{2n-k}) \hookleftarrow I_k = C(B_k)$$

$C(M_{2n})$: Poisson with the trivial Poisson center (M_{2n} = symplectic)

J_{2n-k}: Poisson

$I_k = Z_P(J_{2n-k})$

Quantum

$$\begin{array}{ccccc} A & \hookleftarrow & J & \hookleftarrow & \boxed{I = Z(J)} \\ \| & & \| & & \| \\ C_h(M_{2n}) & & C_h(P_{2n-k}) & & C_h(B_k) \end{array}$$

in any representation V of A joint eigensubspaces of I are finite direct sums of irreducible representations of J.

Examples

Kepler system (classical)

$$C(\mathbb{R}^6) \supset J_5 \supset I_1$$

$J_5 = \langle M_\alpha, A_\alpha, H \mid 2\text{ rel.}\rangle$

I_1: generated by H

max superintegrab.

Hydrogen atom (quantum)

$$\mathrm{Diff}(\mathbb{R}^3) \supset \hat{J}_5 \supset \hat{I}_1$$

$\hat{I}_1$: by $\hat{H}$ gen.

$\hat{J}_5 = \langle \hat{M}_\alpha, \hat{A}_\alpha, \hat{H} \mid 2\text{ rel.}\rangle$

$\hat{J}_5$ – hidden symmetry

$SO(3)$ – manifest symmetry

$V_\ell \otimes V_\ell \simeq V_0 \oplus V_1 \oplus \dots \oplus V_{2\ell}$

Spin Calogero-Moser-Sutherland systems and their generalizations

Hamiltonian reduction

① Spin Calogero-Moser (J. Gibbons, T. Hermsen, 1984; N.R., 2003, 2015) nonspin $H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} \frac{c}{(x_i-x_j)^2}$, $\frac{c}{\sin^2(x_i-x_j)}$ B. Sutherland

Classical: General fact: assume that

- M is a symplectic manifold,
- $G \times M \to M$ Hamiltonian action of a Lie group.

i.e. $X \in \mathfrak{g}$, $\boxed{X \cdot f = \{H_X, f\}}$,

$H_X(x) = \langle X, \mu(x) \rangle$, $\mu : M \to \mathfrak{g}^*$ moment map

Then:

- $\boxed{M/G - \text{Poisson}}$ (if nonsingular), $C(M)^G \subset C(M)$
- $S(\mathcal{O}) \subset M/G$ symplectic leaf, $\mathcal{O} \subset \mathfrak{g}^*$

$\boxed{S(\mathcal{O}) = \mu^{-1}(\mathcal{O})/G}$ $\mu^{-1}(\mathcal{O}) \subset M$ coisotropic

Hamilt. reduction with respect to $\mathcal{O}$

Consider $M = T^*G \simeq \mathfrak{g}^* \times G$ (p, x; q^i)

(right translations)

(i) $\{e_i\}$ – basis in $\mathfrak{g}$, $\pi : G \to \mathrm{Aut}(V)$, $\pi_{ab}(g)$

$\{e^i\}$ – dual in $\mathfrak{g}^*$,

x_i – coord. on $\mathfrak{g}^*$

$$\begin{cases} \{x_i, x_j\} = -C_{ij}^{k} x_k, \quad \{x_i, \pi_{ab}(g)\} = \sum_c \pi_{ac}(e_i)\pi_{cb}(g) \\ \{\pi_{ab}(g), \pi_{cd}(g)\} = 0 \end{cases}$$

$T^*G \simeq \mathfrak{g}^* \times G$

(ii) Left "action" by mult. from the left $G : (\mathfrak{g}^* \times G) \circlearrowleft$

$$\alpha_L(h)(x,g) = (Ad^*_h(x), hg)$$

is Hamiltonian

$$Xf = \frac{d}{dt} f(\alpha_L(e^{-tX})(x,g)) = \{H^L_X, f\}(x,g), \quad X \in \mathfrak{g}$$

$X \in \mathfrak{g}$

$$H^L_X(x,g) = \langle X, x \rangle = \langle X, \mu_L(x,g) \rangle$$

$x \in \mathfrak{g}^*$ $\Rightarrow$

$\boxed{\mu_L(x,g) = x}$; $\boxed{\mu_L : T^*G \simeq \mathfrak{g}^* \times G \to \mathfrak{g}^*}$

is the moment map for the left action

(iii) For the right ~~action~~ the left action by mult. from the right

$T^*G \simeq \mathfrak{g}^* \times G$

$$\alpha_R(h)(x,g) = (x, gh^{-1})$$

$$H^R_X(x,g) = -\langle X, Ad^*_{g^{-1}}(x) \rangle$$

$\boxed{\mu_R : T^*G \to \mathfrak{g}^*}$, $\boxed{\mu_R(x,g) = -Ad^*_{g^{-1}}(x)}$

(iv) The adjoint action T^*G/Ad_G

$$Ad_h : (x, g) \mapsto (Ad_h^*(x), hgh^{-1})$$

$$\boxed{\mu : T^*G \to \mathfrak{g}^*}, \quad \boxed{\mu = \mu_L + \mu_R}$$

(v) Symplectic leaves of $T^*G/_{Ad_G} \simeq (\mathfrak{g}^* \times G)/Ad_G$

$$O \subset \mathfrak{g}^*, \quad O \in \mathfrak{g}^*/G, \quad \boxed{S(O) = \mu^{-1}(O)/Ad_G}$$

$$\boxed{S(O) = \{(x, g) \in \mathfrak{g}^* \times G \mid \boxed{x - Ad_{g^{-1}}^*(x) \in O}\}/Ad_G}$$

$G = SU_n$, by conjugation $g = g' h g'^{-1}$ $\quad x = (x_{ij})$

$i \neq j$ $\quad x_{ij} - h_i x_{ij} h_j^{-1} = \mu_{ij}$ "coordinates" on O, $\boxed{\mu_{ii} = 0}$

x_{ii} – no condition, remaining conjugations with respect to $H \subset SU_n$ (diagonal matrices, Cartan subgroup). $N(H)$

$$\boxed{\text{Functions on } S(O) = \{\text{Functions in } \mu_{ij}, i \neq j, h_i^{\pm 1}, x_{ii}\}^{N(H)}, \quad \mu_{ii} = 0}$$

Hamilt. red. with respect to H-action $r_L + r$

$$\boxed{S(O) \simeq (T^*_{p\ \ x_{ii} = q_i} H \times O//H)/W}, \quad \boxed{N(H) \supset H}$$

$N(H)/H = W$

$W = S_n$

Alg. functions on $S(\mathcal{O})$ =

$= \mathrm{Pol}\left(\dfrac{\mu_{ij}}{1-h_ih_j^{-1}}, h_i^{\pm 1}, x_{ii} \right)^{S_n}$

$\dim(S(\mathcal{O})) = \underset{p,q}{2r} + \underbrace{\dim(\mathcal{O})-2r}_{\dim(\mathcal{O}/\!/H)} = \dim(\mathcal{O})$, $r = n-1$

A superintegrable system on $S(\mathcal{O})$

all maps are Poisson

$$\begin{tikzcd}
T^*G \simeq \mathfrak{g}^* \times G \arrow[r, "\mu_L \oplus \mu_R"] \arrow[d] & \mathfrak{g}^* \tilde{\times}_{\mathfrak{g}^*/G} \mathfrak{g}^* \arrow[r, shift left, "\pi_L"] \arrow[r, shift right, "\pi_R"'] \arrow[d] & \mathfrak{g}^* \arrow[d] \\
T^*G/\mathrm{Ad}_G \arrow[r, "p_1"] & (\mathfrak{g}^* \tilde{\times}_{\mathfrak{g}^*/G} \mathfrak{g}^*)/\mathrm{Ad}^*_G \arrow[r, "p_2"] & \mathfrak{g}^*/\mathrm{Ad}^*_G \\
S(\mathcal{O}) \arrow[u, hook] \arrow[r, "p_1"] & P(\mathcal{O}) \arrow[u, hook] \arrow[r, "p_2"] & B(\mathcal{O}) \arrow[u, hook]
\end{tikzcd}$$

$(x, g) \mapsto (x, -\mathrm{Ad}^*_{g^{-1}}(x)) \overset{\pi_L}{\longmapsto} x$

Poisson   Poisson   triv. Pois.

restrict to $p_1(S(\mathcal{O}))$ $= p_2 p_1(S(\mathcal{O}))$

$\mathfrak{g}^* \times_{\mathfrak{g}^*/G} \mathfrak{g}^* = \{ (x,y) \mid Gx = -Gy \}$ $\quad \in \mathfrak{g}^* \times \mathfrak{g}^*$

- $P(\mathcal{O}) = \{ (\underset{\in \mathfrak{g}^*}{x}, \underset{\in \mathfrak{g}^*}{y}) \mid Gx = -Gy, \ x - y \in \mathcal{O} \}/G$

$$\dim(\mathcal{P}(\mathcal{O})) = \dim(\mathcal{O}) - r \qquad \underline{\underline{u}} \in \mathcal{O}$$

- $B(\mathcal{O}) = \{\mathcal{O}' \mid m(\mathcal{O}', -\mathcal{O}', \mathcal{O}) \neq \emptyset\}$, $\in \mathfrak{g}^*/G$

$$m(\mathcal{O}', -\mathcal{O}', \mathcal{O}) = \{(x,y) \in \mathcal{O}' \times (-\mathcal{O}') \mid x - y \in \mathcal{O}\}/G$$

$$= (\mathcal{O}' \times (-\mathcal{O}') \times \mathcal{O})//G \quad , \quad \text{symplectic}$$

sympl.

Hamiltonian with resp. to the diag. action of G

$$\dim(B(\mathcal{O})) = r$$

- $\boxed{p_2^{-1}(\mathcal{O}') = m(\mathcal{O}', -\mathcal{O}', \mathcal{O})}$, $\boxed{\dim(p_2^{-1}(\mathcal{O}')) = \dim(\mathcal{O}) - 2r}$

$$\mathcal{P}(\mathcal{O}) = \bigsqcup_{\mathcal{O}' \in B(\mathcal{O})} m(\mathcal{O}', -\mathcal{O}', \mathcal{O})$$

$\dim(\mathcal{O}) \geq r$

Poisson commuting Hamiltonians:

$$I = C(B(\mathcal{O})) = C(\mathfrak{g}^*)^G \Big|_{B(\mathcal{O})}$$

Casimir functions on $\mathfrak{g}^*$.

For SU_n: $\boxed{H_k = \frac{1}{k} \mathrm{tr}(x^k)}$ $k = 2, \dots, n$

$\boxed{x_{ii} = p_i}$, $\boxed{x_{ij} = \dfrac{\mu_{ij}}{1 - e^{i(q_i - q_j)}}}$ $h_j = e^{iq_j}$

$$\boxed{H_2 = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} \frac{\mu_{ij}\mu_{ji}}{2\sin^2\left(\frac{q_i-q_j}{2}\right)}}$$

for SU_n, $\mu_{ij}\mu_{ji} = |\mu_{ij}|^2$ like n particles 1D, with "spin variables" μ_{ij}

$\{\mu_{ij}, \mu_{ke}\} = \mu_{ie}\delta_{jk} - \dots$

Rank 1 orbits $\boxed{\mu_{ij} = \overline{\varphi_i}\,\varphi_j - \delta_{ij}\frac{(\overline{\varphi}, \varphi)}{n}}$

$\boxed{\mathcal{O} \text{ is } (2n-2)\text{-dimensional}}$ $\mathcal{O}/\!/H \simeq \text{pnt}$

$2n-2-2(n-1) = 0$

$\boxed{|\mu_{ij}| = c}$. This is the case of the CM nonspin model (Kazhdan-Kostant-Sternberg)

Quantum spin Calogero-Moser system

Quantization :

$$
\begin{tikzcd}
\overset{C(T^*G)}{\mathrm{Diff}(G)} & \overset{C(\mathfrak{g}^* \times_{\mathfrak{g}^*/G} \mathfrak{g}^*)}{U\mathfrak{g} \tilde{\otimes}_{Z(U\mathfrak{g})} U\mathfrak{g}} \arrow[l, hook'] & U\mathfrak{g} \arrow[l, shift right, "L"'] \arrow[l, shift left, "R"] \\
\mathrm{Diff}_G(G) \arrow[u] & \left(U\mathfrak{g} \tilde{\otimes}_{Z(U\mathfrak{g})} U\mathfrak{g}\right)^G \arrow[l] & Z(\mathfrak{g}) \arrow[l, hook'] \arrow[u]
\end{tikzcd}
$$

$x z \otimes y = x \otimes S(z) y$

$z \in Z$

adjoint action

$Z(\mathfrak{g}) = (U\mathfrak{g})^G$

~~$A \supset J \supset I$~~

The Hilbert space

G - compact simple

$$\mathcal{H}_\mu = \langle L_2(G \to V_\mu) \mid f(g h g^{-1}) = \pi_\mu(g) f(h),\ g \in G,$$

$$h \in H \subset G,\ f(h) \in V_\mu[0] \subset V_\mu \rangle \simeq L^2(H \to V_\mu[0])$$

$\mathrm{Diff}(G)$ acts naturally on $\mathcal{H}_\mu$:

$X \in \mathfrak{g} \subset \mathrm{Diff}(G)$

$$(X_L f)(g) = \frac{d}{dt} f(e^{-tX} g)\Big|_{t=0}$$

$$(X_R f)(g) = \frac{d}{dt} f(g e^{tX})\Big|_{t=0}$$

if $f \in \mathcal{H}_\mu$

$$(X_L + X_R) f(g) = \pi_\mu(X) f(g)$$

For sl_n, $\{e_{ij}\}$ - basis $\left(\sum_{i=1}^n e_{ii} = 0\right)$

$$(e_{ij}^R f)(h) = \frac{d}{dt} f(h e^{t e_{ij}})\Big|_{t=0} = -h_i h_j^{-1} e_{ij}^L f(h)$$

$$\Rightarrow \quad (e_{ij}^L - h_i h_j^{-1} e_{ij}^L) f(h) = \pi_\mu(e_{ij}) f(h)$$

$$\boxed{\begin{aligned} &\text{su}_n \quad e_{ij}^L f(h) = \frac{1}{1 - h_i h_j^{-1}} \pi_\mu(e_{ij}) f(h), \\ &e_{ii}^L f(h) = \frac{d}{dt} f(e^{-t e_{ii}} h) = -h_i \frac{\partial f}{\partial h_i}(h) \\ &\qquad\qquad h_j = e^{iq_j} \end{aligned}}$$

Corollary We can compute Casimirs

$$H = C_2 = \underbrace{-\frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial q_i^2} + \frac{1}{4}\sum_{i \neq j} \frac{\pi_\mu(e_{ij} e_{ji})}{\sin^2\left(\frac{q_i - q_j}{2}\right)}}$$

$\mu_{ij}\mu_{ji}$

$$\underline{C_k = \frac{1}{k} \operatorname{tr}\left((e^L)^k\right)}$$

Commutative differential operators.

<u>Superintegrability of quantum</u>

<u>spin CM</u>

$$\begin{tikzcd}
\mathrm{Diff}_G(G) \arrow[d] & (Ug \otimes_{Z(Ug)} Ug)_G \arrow[l, hook'] \arrow[d] & Z(g) \arrow[l, hook'] \arrow[d] \\
A_\mu & J_\mu \arrow[l, hook'] & I_\mu \arrow[l, hook']
\end{tikzcd}$$

$I_\mu = \langle H_2, H_3, \dots \rangle$

<u>Joint spectrum of $\langle H_2, H_3, \dots \rangle$</u>

$$\boxed{\mathcal{H}_\mu = \bigoplus_{\lambda \in D_\mu} \mathcal{H}_\mu[\lambda]}, \quad \boxed{\mathcal{H}_\mu[\lambda] \simeq \mathrm{Hom}_G(V_\lambda \otimes V_\lambda^*, V_\mu)}$$

$$D_\mu = \{\lambda \mid \mathrm{Hom}_G(V_\lambda \otimes V_\lambda^*, V_\mu) \neq \emptyset\}$$

Vectors in $\mathcal{H}_\mu[\lambda]$:

$$\boxed{\Psi_a(g) = \mathrm{Tr}_{V_\lambda}(\pi_\lambda(g)\, a)}, \quad \text{Trace functions}$$

$a : V_\lambda \to V_\lambda \otimes V_\mu$, $a \in \mathrm{Hom}_G(V_\lambda \otimes V_\lambda^*, V_\mu)$

$$\hat{H}_k \Psi_a(g) = c_k(\lambda)\, \Psi_a(g)$$

Proposition $H_\mu(\lambda)$ is irreducible for J_μ.

References

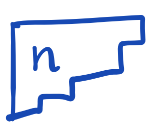
tqfts.com

Special cases:

1) $\mu = m\omega_1$ $\left(\mathrm{Sym}^m(\mathbb{C}^n)\right)$ sl_n ,

$V_\mu[0] \neq \{0\}$ iff $m = nk$,

$$\boxed{H = -\Delta + \sum_{i<j} \frac{2k(k+1)}{\sin^2\left(\frac{q_i - q_j}{2}\right)}},$$

2) $\mu \vdash n$, $\mu =$  n boxes,

In this case

$V_\mu[0]$ = an irreducible S_n-module corresponding to the Young diagram μ.

For $m > n$

$$\boxed{(\mathbb{C}^m)^{\otimes n} \cong \bigoplus_{\mu \vdash n} V_\mu^{SL_m} \otimes V_\mu[0]}$$

$$\mathcal{H}_n^{(m)} = \boxed{L^2\left(h \to (\mathbb{C}^m)^{\otimes n}\right)^{S_n}} = \underline{\bigoplus_\mu V_\mu^{SL_m} \otimes \mathcal{H}_\mu}$$

n Bose particles with <u>m internal degrees of freedom.</u>

$$\boxed{H = -\Delta + \sum_{i<j} \frac{1 + P_{ij}}{\sin^2\left(\frac{q_i - q_j}{2}\right)}}, \quad \checkmark$$

P_{ij} - permutation in $\underline{(\mathbb{C}^m)^{\otimes n}}$. $L = x$

Lax matrix $L = x = \begin{pmatrix} p_i & & \frac{\pi_\mu(e_{ij})}{1 - h_i h_j^{-1}} \\ & p_{i+1} & \\ & & \ddots \end{pmatrix}$, $\frac{dL}{dt} = \underset{\text{quantum.}}{[H, L]} = [L, M]$ matrix

<u>Semiclassical limit</u>

$$\hat{H} = -\hbar^2 \Delta + \sum_{i<j} \frac{\pi_\mu(e_{ij} e_{ji})}{\sin^2\left(\frac{q_i - q_j}{2}\right)},$$

- acts on $\mathcal{H}_\mu$,
- $\mathcal{H}_\mu[\lambda]$ is an eigensubspace with

$$\hat{H}\, \mathcal{H}_\mu[\lambda] = \hbar^2 c_2(\lambda)\, \mathcal{H}_\mu[\lambda]$$

As $\hbar \to 0$, $\mu = \frac{\mu_c}{\hbar}$, $\lambda = \frac{\lambda_c}{\hbar}$

(i) $A_\mu \supset J_\mu \supset I_\mu$

$$\begin{array}{ccccc} A_\mu & \supset & J_\mu & \supset & I_\mu \\ \downarrow{\scriptstyle \hbar\to 0} & & \downarrow{\scriptstyle \hbar\to 0} & & \downarrow{\scriptstyle \hbar\to 0} \\ C(S(\mathcal{O}_{\mu_c})) & \supset & C(P(\mathcal{O}_{\mu_c})) & \supset & C(B(\mathcal{O}_{\mu_c})) \end{array}$$

$\hbar^2 c_2(\lambda) \to (\lambda_c, \lambda_c)$

(ii) $\dim(\mathcal{H}_\mu[\lambda]) = \dim(\operatorname{Hom}_G(V_\lambda \otimes V_\lambda^*, V_\mu)) =$

$= \hbar^{-\frac{n}{2}} \operatorname{Vol}(\mathcal{M}(\mathcal{O}_{\lambda_c}, \mathcal{O}_{-w_0(\lambda_c)}, \mathcal{O}_{\mu_c}))(1+O(\hbar))$

$n = \dim(\mathcal{M}(\mathcal{O}_{\lambda_c}, \mathcal{O}_{-w_0(\lambda_0)}, \mathcal{O}_{\mu_c}))$

$$\mathcal{M}(\mathcal{O}_s, \mathcal{O}_t, \mathcal{O}_u) = (\mathcal{O}_s \times \mathcal{O}_t \times \mathcal{O}_u) /\!\!/ G =$$

$$= \{(x, y, z) \mid x \in \mathcal{O}_s, y \in \mathcal{O}_t, z \in \mathcal{O}_u, \; x+y+z = 0\} / G$$

In a Liouvill integrable system the multiplicity of an eigensubspace is stable as $\hbar \to 0$ and

$$\mathrm{mult}(I_i \psi = E_i \psi) \overset{\hbar \to 0}{=} \#\{\text{connected component of the corresponding level surface of classical integrals}\}.$$

Next time:

- N-spin CM type systems.
- 2D QCD revisited again.