# Superintegrable Systems. I.

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## Plan:

1. Superintegrable Hamiltonian systems

2. Examples

3. Quantum s. int. systems

3. Superintegrable systems on moduli spaces of G-flat connections on a surface

(based on joint work with S. Artamonov and J. Stokman)

# Superintegrable systems

Degenerate integrability, noncommutative integrability; Fock, Pauli; Smorodinsky, Winterniz.... Nekhoroshev, Mischenko-Fomenko...

$(M_{2n}, \omega)$ - symplectic manifold $\{p_i, q^i\}$

$\omega = \sum_i dp_i \wedge dq^i$

$\{p_i, q^j\} = \delta_i^j$

$A = C^\infty(M_{2n})$, $\boxed{H \in A}$

Poisson bracket

$Z(H, A) = \{ a \in A \mid \{H, a\} = 0 \}$

$(A, \{\cdot,\cdot\})$ Poisson algebra

centralizer of $H$ in $A$

conserv. laws for the Ham. mech. generated by $H$.

**Def.** Hamiltonian flow generated by $H$ is superintegrable if

(i) $\exists\ J \subset A$ of rank $\underline{2n-k}$, i.e.
Poisson subalgebra

$\exists\ J_1, \dots, J_{2n-k} \in J$,

a maximal independent set

(ii) Poisson center of $J$ (Poisson alg.)

$$Z(J) = \{ a \in J \mid \{a, b\} = 0, \ \forall b \in J \}$$

has rank $k$, i.e. contains maximum $k$ independent functions $I_1, \dots, I_k$ and $H \in Z(J)$

$\boxed{\text{Liouville case } k = n}$ $J = I$

Thus algebraically a superintegrable system is

$$\underset{2n}{A} \supset \underset{2n-k}{J} \supset \underset{k}{I = Z(J)}$$

Poisson subalgebras

$2n-k$: integrals of motion, conserved quantities

$k$: Poisson commuting Hamiltonians

# Geometrically

$$\begin{tikzcd}
\mathrm{Spec}(A) = M_{2n} \arrow[d, "p_1"] \\
\mathrm{Spec}(J) = \mathcal{P}_{2n-K} \arrow[d, "p_2"] \\
\mathrm{Spec}(I) = B_K
\end{tikzcd}$$

symplectic

$K$-dim generic isotropic fibers

$\{\varphi_i\}$

$J_1 \dots J_{2n-K}$ local coord.

generic fibers are $(2n-2K)$-dim symplectic leaves

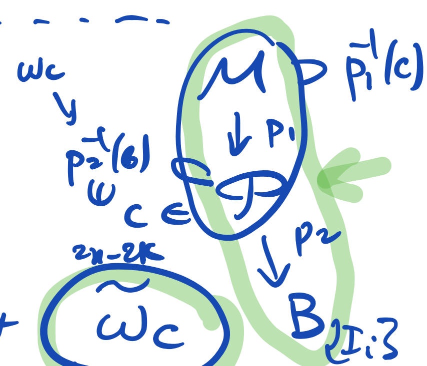
$I_1, \dots, I_K$ local coord. on $B_K$

$K < n$

$K = n$ Liouville case

**Theorem** (Nekhoroshev)

- Flow lines of $I_1, \dots, I_K \in I$ generate affine coordinates $\varphi_1, \dots, \varphi_K$ angle variables on level surfaces of $J_1 \dots J_{2n-K}$ (on $p_1^{-1}(c)$)




- In a nbd of $p_1^{-1}(c)$

$$\overset{2n}{\omega} = \sum_{i=1}^{K} \overset{2K}{dI_i \wedge d\varphi_i} + \overset{2n-2K}{\widetilde{\omega}_c}$$

$\tilde{\omega}_c$ = sympl. form on $p_2^{-1}(p_2(c))$

- Each connected component of $p_1^{-1}(c)$ is $\simeq$ $\mathbb{R}^{\ell} \times \mathbb{T}^{k-\ell}$

- Flow lines of any $H \in Z(J)$ are linear in $\varphi_1 .. \varphi_n$:

$$\varphi_i(t) = \omega_i(H, c)\, t + \varphi_i(0)$$

Refinement:

(k=n)

Liouville integrable on $M_{2n}$

n-dim invariant tori

$A_{2n} \supset J_n = I_n$

can be superintegrable

(k<n)

Superintegrable on $M_{2n}$

k-dim invariant tori

$A_{2n} \supset J_{2n-k} \supset I_k$

can be integrable

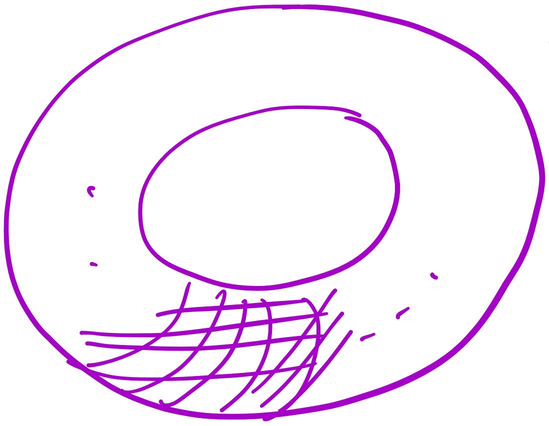
$I_n$ Poisson commut. but central subalgebra

Refinement: $A_{2n} \supset J_{2n-k} \supset J_n \supset I_k$

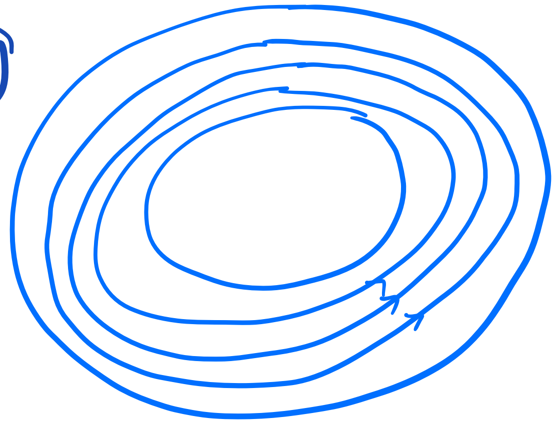
do not Poisson commute

$Z(J)$

$M_4$



$I_1, I_2$



$J_1, J_2, J_3$

2-dim torus
densely covered by flow lines of $I_1, I_2$
Integrable, possibly non periodic trajectories

2-dim torus covered by 1-dim tori, level curves of $J_1, J_2, J_3$ (flow lines of $I_1$)
Superintegrable, periodic trajectories

Refinement, when exists

Examples:

1) Kepler system, $M = \underset{p}{\mathbb{R}^3} \times \underset{q}{\mathbb{R}^3} = T^*\mathbb{R}^3$

$\omega = \sum_{i=1}^{3} dp_i \wedge dq^i$,

$$\boxed{H = \frac{1}{2}\vec{p}^2 - \frac{\gamma}{|q|}},$$

Momenta:

$$\vec{M} = \vec{p} \times \vec{q} \quad ✓ \qquad 6$$

Lenz vector:

$$\vec{A} = \vec{p} \times \vec{M} + \gamma \frac{\vec{q}}{|\vec{q}|},$$

Poisson brackets:

$$\begin{cases} \{M_i, M_j\} = \varepsilon_{ijk} M_k, \ \{M_i, A_j\} = \varepsilon_{ijk} A_k, \\ \{A_i, A_j\} = -2H\, \varepsilon_{ijk} M_k, \quad \begin{matrix} H>0 \\ -2H<0 \end{matrix} \quad 6+1=7 \\ \boxed{\{H, M_i\} = \{H, A_i\} = 0} \end{cases}$$

Relations:

$$(\vec{M}, \vec{A}) = 0, \quad \vec{A}^2 = \gamma^2 - 2\vec{M}^2 \cdot H$$

$M_6 \to J_5 \to B_1$

$$C(T^*\mathbb{R}^3) \supset J_5 \supset I$$

$p \quad q$

$$\text{(6)} \qquad \text{(5)} \qquad \text{(1)}\ I = \langle H \rangle$$

$$J_5 = \langle M_i, A_i, H \rangle / \text{Relations}$$

$$\begin{cases} \text{Poisson subalgebra,} \\ I = Z(J_5) \end{cases}$$

Geometrically

$$\underset{\approx \mathbb{R}^6}{M_6} \xrightarrow[p_1]{} \mathcal{P}_5 \xrightarrow[p_2]{} \underset{}{B_1} \approx \mathbb{R}$$

possible values of $H$

maximally superintegrable

Fibers of $p_2$ $\left(p_2^{-1}(E)\right)$ : sympl. leaves of $\mathcal{P}_5$

$\mathcal{O} \cong S^2 \times S^2 \subset so(4)^*$ $\qquad$ $T^*S^2 \subset E(3)^*$ $\qquad$ $\mathcal{O} \subset so(3,1)^*$

$E<0 \qquad E=0 \qquad E>0 \qquad E$

hidden $so(3,1)$ symmetry

Liouville integrable system: $A_z, M_z, H$

# Quantum superintegrable systems

1) $(M, \omega)$ - symplectic $\begin{array}{c} M_{2n} \\ \downarrow \\ B_n \end{array}$ integrable system.

$A_h = C_h(M)$ quantization of $C(M)$

associative algebra quantizing $C(M)$

in a sense that "$A_h \to C(M)$" as $h \to 0$ as a commutative algebra

and

$$\{f, g\} = \lim_{\hbar \to 0} \frac{\hat{f}\hat{g} - \hat{g}\hat{f}}{\hbar}$$

Def. A maximal commutative subalgebra $I_h \subset A_h$ is a quantum integrable system quantizing $\begin{array}{c} M_{2n} \\ \downarrow \\ B_n \end{array}$ if

- $I_h$ is a deformation of $I = C(B_n)$
- In every irreducible representation of $A_h$, the joint spectrum of $I_h$

has finite multiplicities.

(quantum integrable systems of finite type. $\begin{cases} A_h = \mathrm{Diff}(Q_n), \ M_{2n} = T^*Q_n \\ I_h \subset A_h \ n \text{ comm. d. oper.} \end{cases}$)

2) $(M, \omega)$ - symplectic, $M_{2n} \to P_{2n-k} \to B_k \ (*)$

is a superintegrable system.

Let $A_h$ be a quantization of $C(M)$.

<u>Def.</u> A quantization of the superintegrable system $(*)$ is

$$A_h \supset J_h \supset I_h$$

gen. of symmetries $J_1, \ldots, J_{2n-k}$ $\quad \{\hat{I}_1, \ldots, \hat{I}_k\}$

$\{\hat{J}_i, \hat{I}_j\} = 0$

- $I_h$ = commutative of rank $k$, deforming $I = C(B_k)$
- $J_h$ = associative, quantizing $C(P_{2n-k})$
- $I_h = Z(J_h)$
- In every representation of $A_h$

joint eigensubspaces of $I_\hbar$ are isomorphic to finite sums of irreducible $J_\hbar$-modules.

Hydrogen atom (quantum Kepler system):

$$A_\hbar \qquad\qquad\qquad\qquad\qquad Z(\hat{J}_\gamma)$$

$$\mathrm{Diff}(\mathbb{R}^3)_{\mathbb{C}} \supset \hat{J}_\gamma \supset \hat{I} = \mathbb{C}\hat{H}$$

quantum algebra of observables

$$\hat{H} = -\Delta - \frac{\gamma}{|q|},$$

Subalgebra generated by $\hat{M}_a$, $\hat{A}_a$, $\hat{H}$ with relations

7 generators

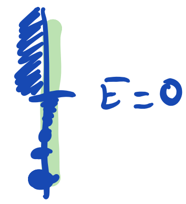
2 relations

$$[\hat{M}_a, \hat{M}_b] = i\varepsilon_{abc}\hat{M}_c, \quad [\hat{M}_a, \hat{A}_b] = i\varepsilon_{abc}\hat{A}_c$$

$$[\hat{M}_a, \hat{H}] = [\hat{A}_a, \hat{H}] = 0, \quad + 2 \text{ relations}$$

$$\mathrm{Spec}(\hat{H}) = \{\text{discrete}\} \sqcup \underline{\{E=0\}} \sqcup \underline{\{E>0\}}$$

$L_2(\mathbb{R}^3)$




$$E_n = -\frac{C}{n^2\hbar^2}, \quad n=1,2,\dots$$

$$\mathrm{Eigenspace}(E_n) \simeq V_{n-1}\otimes V_{n-1}, \quad \dim(V_\ell)=\ell+1$$

$n^2$ – degenerate.

- $\hat{M}_a$ acts diagonally, the action is reducible

$$V_{n-1}\otimes V_{n-1} \simeq V_0 \oplus V_2 \oplus \cdots$$

orbitals

This is a symmetry (not complete) of $\hat{H}$.

- $\hat{J}_5$ acts irreducibly on $V_{n-1}\otimes V_{n-1}$

This is the true symmetry of $\hat{H}$.

As $\hbar \to 0$, $n\to\infty$, $n\hbar = n_c$ is fixed

$$\underbrace{\dim(\mathrm{Eigenspace}(E_n))}_{\text{"}n^2} \propto \hbar^{-2} = \hbar^{-\frac{\dim(p_2^{-1}(E_{n_c}))}{2}}$$

$$\dim(p_2^{-1}(E)) = 4$$

geometric quantization