Lect 5

Last time: - the decomposition of $V_{\ell_1} \otimes V_{\ell_2}$ into irreducibles

- Quantum Weyl group

Quantum Weyl group (cont'd)

Def. The algebra generated by E, F, H and w where E, F, H generate $U_h sl_2$, $\theta(a) = waw^{-1}$ and

$$\Delta w = (w \otimes w) R = R_{21} (w \otimes w)$$

is called the Chevalley extended quantum sl_2, we will denote it $\widetilde{U_h sl_2}$.

(uniqueness of w ....)

Theorem In $\widetilde{U_h sl_2}$ the following holds

$$\varepsilon(w) = 1, \quad S(w) w = S(u)^{-1}$$

where $u = \sum_i S(b_i) a_i$, $R = \sum_i a_i \otimes b_i$

Proof. $w = (\varepsilon \otimes id)\Delta w = (\varepsilon \otimes id)(w \otimes w) R =$

$= \varepsilon(w) w$. Here we used the definition of the counite and $(\varepsilon \otimes id) R = 1$

Now we will use the property of antipode $m(S \otimes id)\Delta(a) = \varepsilon(a) \cdot 1$, and apply it to w:

$$1 = m(S \otimes id)\Delta w = m(S \otimes id)(b_i w \otimes a_i w) =$$

$$= S(w) S(b_i) a_i w = S(w) u w =$$

$$= u S^{-1}(w) w$$

Here we used the property of u, $S^2(a) = u a u^{-1}$.

Thus
$$S^{-1}(w) w = u^{-1}$$
which proves the second statement.

Remark. Recall that if G is a Lie group, it acts naturally on its Lie algebra via adjoint action and therefore its acts naturally on $U\mathfrak{g}$. Thus, any subgroup of G also acts naturally on $U\mathfrak{g}$

Let G be a simple Lie group and $H \subset G$ is its Cartan subgroup. For $G = SL_2$, the usual choice of H is diagonal matrices, i.e. $H = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \right\}$. The normalizer of H, $N(H)$ is a subgroup in G. For SL_2

$$N(H) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix} \middle| \, x, y \in \mathbb{C}^* \right\}$$

$H \subset N(H)$ is a normal subgroup.

$W = N(H)/H$ is the Weyl group of $\mathfrak{g}$, it acts naturally, by reflections, on the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ and on its dual space. For sl_2:

$$W = \{ e, w \mid w^2 = e \}$$

The normalizer $N(H)$ acts naturally on $\mathfrak{g}$, but not W. However, we can choose representatives $\dot{e}, \dot{w} \in N(H)$ of classes $e, w \in W$, for example as

$$\dot{e} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \dot{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in PSL_2$$

They act on $U\mathfrak{sl}_2$, $\dot{e}$ acts by the identity and $\dot{w}$ acts as

$$E \mapsto -F, \quad F \mapsto -E, \quad H \mapsto -H$$

(by conjugation on 2×2 matrices)

When $q=1$ ($h=0$) the algebra $\widetilde{U_h \mathfrak{sl}_2}$ is the completion of $U\mathfrak{sl}_2$ by the element

$$\dot{w} = \exp(-F)\exp(E)\exp(-F)$$

This element is a representative of $w \in W$, $w = \dot{w}H$. In PSL_2 this the following product of triangular matrices

$$\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{c}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}=\begin{pmatrix}1 & -\frac{1}{c}\\ c & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}=\begin{pmatrix}0 & -\frac{1}{c}\\ c & 0\end{pmatrix}$$

for $c=1$. In this case everything follows from simple identities in $N(H)$:

$$\begin{pmatrix}0 & -\frac{1}{c}\\ c & 0\end{pmatrix}^2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},$$

$$\begin{pmatrix}0 & -\frac{1}{c}\\ c & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{c}\\ -c & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & \frac{1}{c}\\ -c & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ -c^2 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & -\frac{1}{c}\\ c & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{c}\\ -c & 0\end{pmatrix}=\begin{pmatrix}-\frac{1}{c} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{c}\\ -c & 0\end{pmatrix}=\begin{pmatrix}0 & -\frac{1}{c^2}\\ 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & -\frac{1}{c}\\ c & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \frac{1}{c}\\ -c & 0\end{pmatrix}=\begin{pmatrix}0 & \frac{1}{c}\\ c & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{c}\\ -c & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

Irreducible finite dimensional representations of $\widetilde{U_h sl_2}$.

Here we will find how w acts on irreducible representations of $U_h sl_2$. Because $wHw^{-1}=-H$ it acts on the weight basis $\{f_m\}$ in V_ℓ as

$$\pi(w)f_m = \alpha_m f_{-m}$$

for some nonzero α_m.

We should have

$$w E w^{-1} = -F, \quad w F w^{-1} = -E$$

Applying these identities to weight vectors we obtain:

$$\pi(w)\sigma(E) f_m = [\ell - m]\, \alpha_{m+1}\, f_{-m-1}$$

and

$$\sigma(\ )\pi(w) f_m = \alpha_m [\ell - m]\, f_{-m-1}$$

Thus

$$\alpha_m = c_\ell (-1)^{\ell - m}, \qquad (*)$$

for some nonzero c_ℓ.

To determine c_ℓ consider $U_h sl_2$ invariant mapping

$$\pi_0 : V_\ell \otimes V_\ell \to \mathbb{C}$$

<u>Lemma</u> $\pi_0(f_m \otimes f_{m'}) = \delta_{m,-m'}\, \beta_m$

where

$$\beta_m = (-1)^{\ell - m}\, e^{\frac{h}{2}(\ell - m)(\ell + m - 1)}\, \frac{[\ell + m]!\,[\ell - m]!}{[2\ell + 1]!}\, a_\ell$$

and a_ℓ is an arbitrary nonzero.

Proof. Since Π_0 is invariant and 1-dimensional representation has weight zero, the r.h.s. should be proportional to $\delta_{m,-m'}$.

Also

$$\pi_0\Big(Ef_m \otimes e^{\frac{hH}{2}} f_{m'} + f_m \otimes \overset{''\ [l-m'] f_{m'+1}}{E f_{m'}}\Big) = 0$$

$$\Pi_0\Big(Ff_m \otimes f_{m'} + e^{-\frac{hH}{2}} f_m \otimes F f_{m'}\Big) = 0$$

These identities imply

$$[l-m]\,\beta_{m+1}\, e^{-h(m+1)} = -[l+m+1]\,\beta_m, \quad \text{i.e. } \frac{\beta_m}{\beta_{m-1}} = -\frac{[l+m]}{[l-m+1]}\, q^{m+1}$$

From here

$$\beta_m = (-1)^{l-m} e^{-h(m+m-1+\dots+(-l+1))}\, \text{const}\cdot \frac{[l+m][l+m-1]\dots[l-l+1]}{[l-m+1][l-m+2]\dots\dots[l-(-l+1)]}$$

$$= (-1)^{l-m}\, e^{\frac{h(l-m)(l+m-1)}{2}}\, \frac{[l+m]!\,[l-m]!}{[2l+1]!}\, a_l$$

∎

Consider the action of w on this tensor product since $\Delta w = (w \otimes w) R$ and $\varepsilon(w) = 1$, we have

$$\pi_0\big((w\otimes w)R(e_\ell\otimes e_{-\ell})\big) = \pi_0(e_\ell\otimes e_{-\ell})$$

From the explicit structure of R:

$$R(e_\ell\otimes e_{-\ell}) = \exp\left(\tfrac{h}{4}H\otimes H\right)e_\ell\otimes e_{-\ell} = e^{-h\ell^2}e_\ell\otimes e_{-\ell}$$

Thus, $\pi_0(we_\ell\otimes we_{-\ell}) = e^{h\ell^2}\,\pi_0(e_\ell\otimes e_{-\ell})$

from $(\star)$ we have $we_\ell = c_\ell$, $we_{-\ell} = (-1)^{2\ell}c_\ell$

Thus

$$c_\ell^2 = (-1)^{2\ell}e^{h\ell^2}$$

This determines c_ℓ up to a sign.

Dual representations

Let (V,π) be a representation of a Hopf algebra A and $\{e_i\}$ is a basis in V.

Right dual to (V,π) is defined as $(V^*, \pi^\vee)$ s.t. $\pi^\vee(a) = \pi(S(a))^*$. Let $\{e^i\}$ be a basis in V^* dual to $\{e_i\}$. If

$$\pi(a)e_i = \sum_j \pi_i^j(a)\, e_j, \quad \text{then}$$
$$\pi^\vee(a)e^i = \sum_j \pi_j^i(S(a))e^j.$$

Now let us find how $U_h sl_2$ acts on the dual basis $\{f^m\} \in (V_\ell)^*$ to a weight basis $\{f_m^\ell\}$ in V_ℓ.

$$\pi^\vee(H)f^m = -2m f^m, \quad \pi^\vee(E)f^m = -e^{-h(m-1)}[\ell-m+1]f^{m-1},$$
$$\pi^\vee(F)f^m = -e^{hm}[\ell+m+1]f^{m+1},$$

As for the Lie algebra sl_2, finite dimensional $U_h sl_2$-modules are self-dual

<u>Thm</u>. The linear map $\varphi_\ell : (V_\ell)^* \simeq V_\ell$ acting on the dual weight basis as

$$\varphi_\ell(f^m) = c(-1)^{\ell-m} q^{-\frac{(\ell-m)(\ell+m-1)}{2}} \frac{[2\ell]!}{[\ell+m]![\ell-m]!} f_{-m}$$

is an isomorphism of $U_h sl_2$-modules.

<u>Proof</u>. Since φ is $\langle H \rangle$-linear

$$\varphi(f^m) = \varphi_m f_{-m}$$

Now,

$$\varphi(\pi^\vee(E) f^m) = -q^{-(m-1)}[\ell-m+1]\,\varphi(f^{m-1}) = -q^{-(m-1)}[\ell-m+1]\,\varphi_{m-1} f_{-m+1}$$

on the other hand

$$\varphi(\pi^\vee(E) f^m) = \pi(E)\varphi(f^m) = \varphi_m [\ell+m] f_{-m+1}$$

i.e. $\varphi_m [\ell+m] = -q^{-(m-1)}[\ell-m+1]\,\varphi_{m-1}$

$$\varphi_m = \left(-q^{-(m-1)}\right)\left(-q^{-(m-2)}\right)\cdots\left(-q^{-\ell+1}\right)\frac{[\ell-m+1][\ell-m+2]\cdots[2\ell]}{[\ell+m][\ell+m-1]\cdots[1]}\varphi_{-\ell}$$

$\blacksquare$

The bilinear form $(x,y) = \langle \varphi^{-1}(x), y \rangle$ is a $U_h sl_2$ invariant mapping:

$$(\cdot,\cdot) : V_\ell \otimes V_\ell \to \mathbb{C}$$

Note. We can fix $\varphi_{-\ell}$ such that

$$\varphi_m^{-1} = \begin{bmatrix} \ell & \ell & 0 \\ m & -m & 0 \end{bmatrix}$$

Graphical calculus

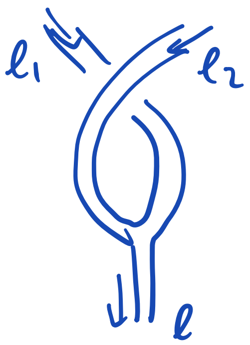
Now we can add to our graphical calculus elements:

with

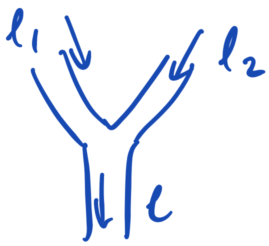
$F($  $) = \pi_\ell(w)$, $\pi_\ell(w)^2 = q^{\frac{c_\ell}{2}}$, $c_\ell = \ell^2 + \ell$

$F($  $) = \varphi_\ell : V_\ell^* \to V_\ell$

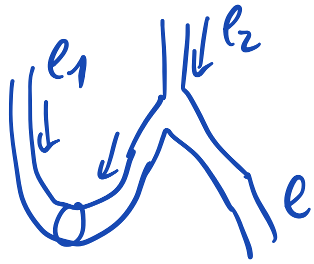
Relations:



$$= (-1)^{\ell_1+\ell_2-\ell} \, q^{\frac{c_\ell - c_{\ell_1} - c_{\ell_2}}{2}}$$



$$=(-1)^{\ell+\ell_1-\ell_2} \ \ldots$$

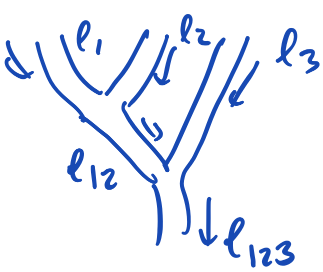


<u>q-6j symbols</u>

Bases in $V_{\ell_1} \otimes V_{\ell_2} \otimes V_{\ell_3}$:

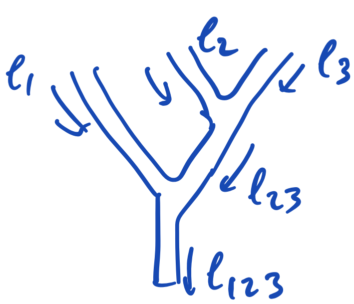
(i) monomial $e^{\ell_1}_{m_1} \otimes e^{\ell_2}_{m_2} \otimes e^{\ell_3}_{m_3}$

(ii)



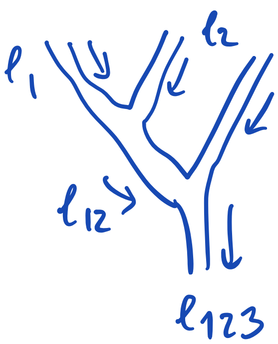
$(V_{\ell_1} \otimes V_{\ell_2}) \otimes V_{\ell_3}$

(iii)

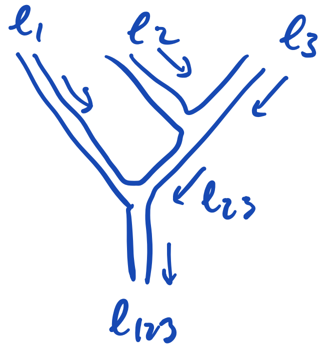


$V_1 \otimes (V_2 \otimes V_3)$

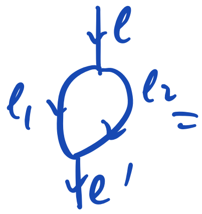
Linear transformation between (iii) & (ii):



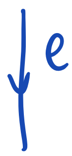
$$= \sum_{l_{23}} \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix}$$



From the relation

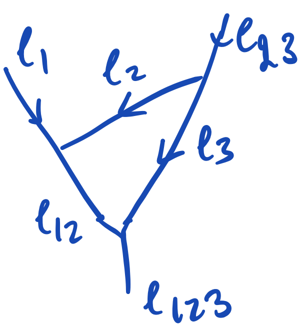


$=$

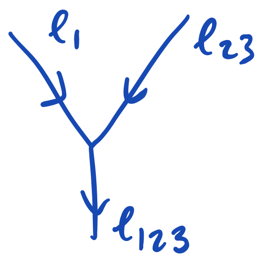


$\delta_{l l'}$

we immediately have:

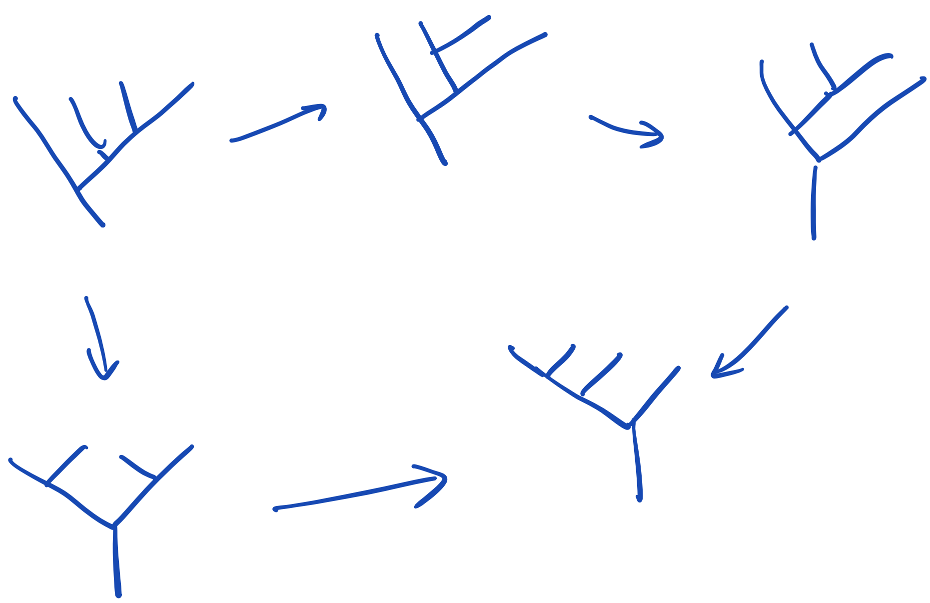


$$= \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix}$$



This allows to compute q-6j as ${}_4\varphi_3$ function [KR]

Pentagon relation for q-6j symbol (Biedenharn - Elliot identity):



This is the associativity constraint for the category $U_h sl_2$ with the following tensor product. Let $V = \bigoplus_e H_e^V \otimes V_e$

$W = \bigoplus_e H_e^W \otimes V_e$, $H_e^W = \mathrm{Hom}(V_e, V)$

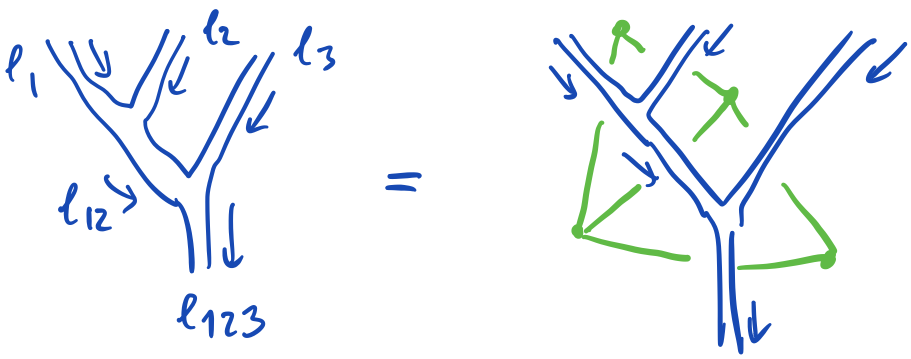
$$V \tilde{\otimes} W = \bigoplus_{e, e', e''} H_e^V \otimes H_{e'}^W \otimes \mathrm{Hom}(V_{e''}, V_e \otimes V_{e'}) \otimes V_{e''}$$

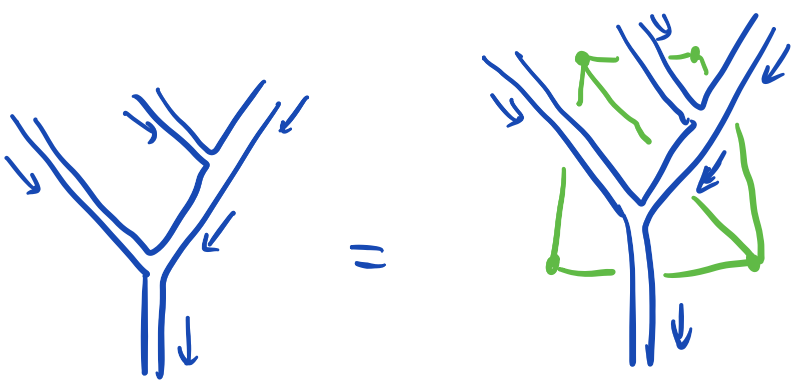
Clearly $(U_h sl_2, \otimes) \simeq (U_h sl_2, \tilde{\otimes})$

3D graphical calculus of tetrahedra

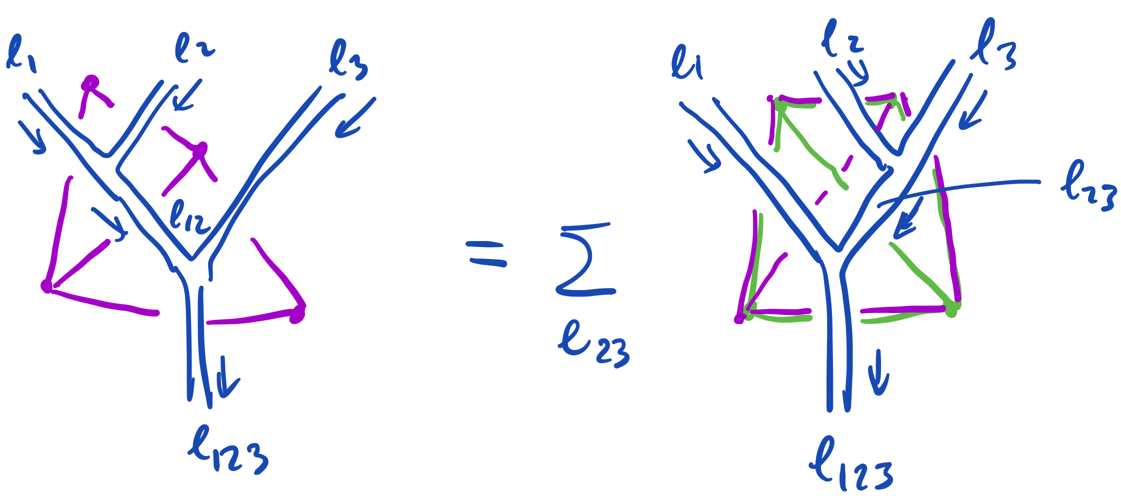
Now, let us interprete q-6j symbols

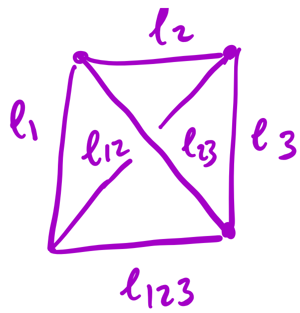
as tetrahedra, i.e. we will lift our graphical caclus to 3D.





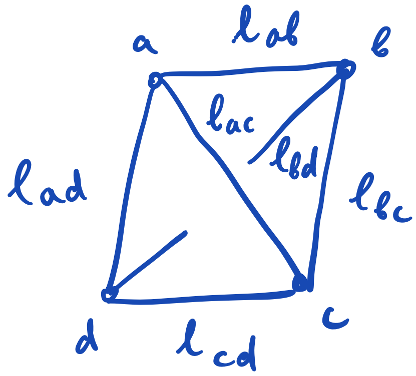
The identity!





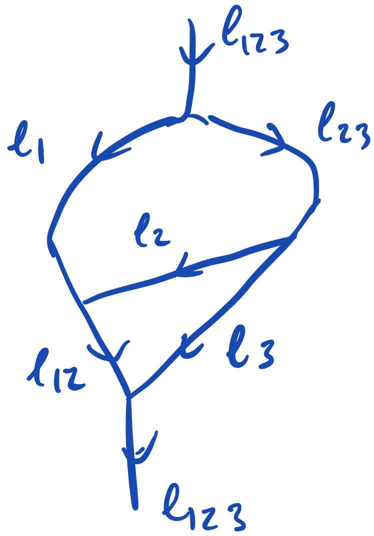
$$\sim \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix}$$

or



$$\sim \begin{Bmatrix} l_{ad} & l_{ab} & l_{bd} \\ l_{bc} & l_{cd} & l_{ac} \end{Bmatrix}$$

Dual picture:

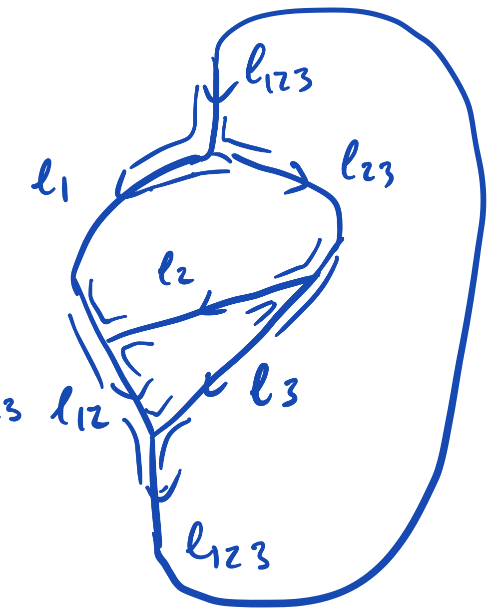


$$= \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix} \downarrow l_{123}$$

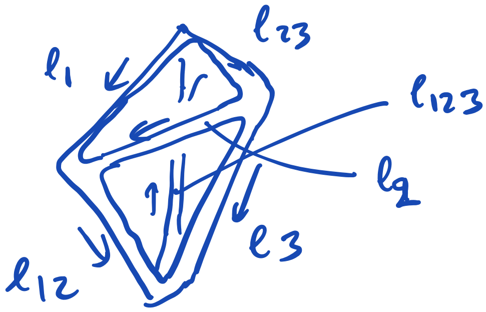
We assume the blackboard framing.

Using the orthogonality of 3j symbols

$$\begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix} = \frac{1}{\bigcirc_{l_{123}}}$$



$$= \frac{1}{\bigcirc}$$



dual tetrahedrons