Lecture 4

Last time:
- $U_h sl_2$
- Its finite dimensional irreducible representations

The decomposition of the tensor product

Let V_{ℓ_1} and V_{ℓ_2} be two finite dimensional $U_h sl_2$-modules. Their tensor product is finite dimensional and therefore is isomorphic to a direct sum of V_ℓ's. It turns out that

$$(*) \quad V_{\ell_1} \otimes V_{\ell_2} \simeq V_{\ell_1+\ell_2} \oplus V_{\ell_1+\ell_2-1} \oplus \cdots \oplus V_{|\ell_1-\ell_2|}$$

For example, the irreducible submodule with the highest weight $\ell_1+\ell_2$ is easy to construct:

- $e_{\ell_1}^{\ell_1} \otimes e_{\ell_2}^{\ell_2}$ is a highest weight vector

  (of weight $\ell_1+\ell_2$) in this tensor product, i.e. $H(e_{l_1}^{l_1} \otimes e_{l_2}^{l_2}) = (l_1+l_2)\, e_{l_1}^{l_1} \otimes e_{l_2}^{l_2}$ and any other highest weight vector is its multiple.

- The action of F on this vector generate the subrepresentation $V_{l_1+l_2} \subset V_{l_1} \otimes V_{l_2}$. It is easy to check that $F^{l_1+l_2+1}(e_{l_1}^{l_1} \otimes e_{l_2}^{l_2}) = 0$.

- The weight subspace of weight l_1+l_2-1 in this tensor product is two dimensional with the monomial basis $e_{l_1-1}^{l_1} \otimes e_{l_2}^{l_2}$ and $e_{l_1}^{l_1} \otimes e_{l_2-1}^{l_2}$. Vector

$$F(e_{l_1}^{l_1} \otimes e_{l_2}^{l_2}) = e_{l_1-1}^{l_1} \otimes e_{l_2}^{l_2} + e^{-h l_1} e_{l_1}^{l_1} \otimes e_{l_2-1}^{l_2}$$

is in the subspace of weight l_1+l_2-1 of $V_{l_1+l_2} \subset V_{l_1} \otimes V_{l_2}$. We can choose the second linearly independent vector in this weight subspace as a

highest weight vector of weight $\ell_1+\ell_2-1$.
The equation

$$E\left(a\, e^{\ell_1}_{\ell_1-2}\otimes e^{\ell_2}_{\ell_2} + b\, e^{\ell_1}_{\ell_1}\otimes e^{\ell_2}_{\ell_2-2}\right)=0$$

Fixes a and b uniquely up to a common multiple.

<u>HW</u>. Find $a:b$

Continuing this process one obtains an explicit decomposition of the tensor product into irreducible components.

To compute the decomposition explicitly, it is convenient to change the weight basis in V_ℓ to $f^\ell_m = b_m e^\ell_m$ with $b_m = [\ell+m]\ldots$

$\ldots[1]$, then

$H f^\ell_m = 2m f^\ell_m$, $F f^\ell_m = [\ell+m] f^\ell_{m-1}$, $E f^\ell_m = [\ell-m] f^\ell_{m+1}$

<u>Theorem</u> [KR] Embeddings $\varphi_\ell : V_\ell \hookrightarrow V_{\ell_1}\otimes V_{\ell_2}$ in the decomposition $(*)$ in terms of weight

basis f^{ℓ}_{m} are given by formulae:

$$\varphi^{z}_{\ell}(f^{\ell}_{m}) = \sum_{m_1+m_2=m} \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{bmatrix}^{z_\ell} f^{\ell_1}_{m_1} \otimes f^{\ell_2}_{m_2} \quad (*)$$

where

$$\begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{bmatrix}^{z_\ell} = z_\ell \, (\text{const}\ldots) \; {}_3\varphi_2\left(\begin{matrix} q^{-n}, ab\, q^{x+1}, q^{-x} \\ aq, q^{-N} \end{matrix} ; q, q\right)$$

Here $n = j - m$, $a = q^{j_2 - j + m_1}$, $b = q^{j_1 - j_2 + m}$, $x = j_1 - m_1$

$N = j + j_2 - m_1$, $q = e^{h}$ and $(\text{const}\ldots)$ is

computed in [KR] (h.w. redo it).

$${}_{p+1}\varphi_p\left(\begin{matrix} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} ; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{p+1}; q)_k}{(b_1, \ldots, b_p; q)_k} z^k,$$

$$(a;q)_k = \prod_{i=0}^{k-1} (1 - a q^i), \quad (a_1, \ldots, a_p; q)_k = \prod_{j=1}^{p} (a_j; q)_k$$

and z_ℓ are arbitrary nonzero scalars.

Remark. The decomposition $(*)$ means that $\mathrm{Hom}(V_\ell, V_{\ell_1}\otimes V_{\ell_2})$ are 1-dimensional for $\ell_1+\ell_2 \geq \ell \geq |\ell_1-\ell_2|$ and zero otherwise. Different choices of z_ℓ correspond to different choices of the basis in $\mathrm{Hom}(V_\ell, V_{\ell_1}\otimes V_{\ell_2})$.

The idea of the proof. Applying $\Delta(E)$, $\Delta(F)$ to both sides of $(*)$ we obtain:

$$[\ell-m]\begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m+1 \end{bmatrix} = [\ell_1-m_1+1]\, q^{m_2} \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1-1 & m_2 & m \end{bmatrix} +$$

$$+ [\ell_2+m_2+1] \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2-1 & m \end{bmatrix},$$

$$[\ell-m]\begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m & 1 \end{bmatrix} = [\ell_1+m_1+1] \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1+1 & m_2 & m \end{bmatrix}$$

$$+ [\ell_2+m_2+1]\, q^{-m_1} \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2+1 & m \end{bmatrix},$$

Similarly, the following holds

Theorem [KR] The projection $\pi_\ell : V_{\ell_1}\otimes V_{\ell_2} \to V_\ell$

$$\pi_\ell^{z}(f^{\ell_1}_{m_1}\otimes f^{\ell_2}_{m_2}) = \begin{bmatrix}\ell_1 & \ell_2 & \ell\\ m_1 & m_2 & m_1+m_2\end{bmatrix}^{z_\ell} f^{\ell}_{m_1+m_2} \qquad (**)$$

is a $U_h sl_2$ - invariant linear mapping and

(i) $$\pi^{z}_\ell \circ \varphi^{z^{-1}}_\ell = id_{V_\ell}$$

(ii) $$\sum_{\ell} \varphi_\ell^{z_\ell}\circ \pi_\ell^{z_\ell^{-1}} = id_{V_{\ell_1}\otimes V_{\ell_2}}$$

or, explicitly:

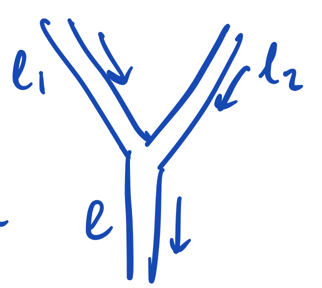
$$\sum_{m_1,m_2} \begin{bmatrix}\ell_1 & \ell_2 & \ell\\ m_1 & m_2 & m\end{bmatrix}^{z_\ell} \begin{bmatrix}\ell_1 & \ell_2 & \ell'\\ m_1 & m_2 & m'\end{bmatrix}^{z_{\ell'}^{-1}} = \delta_{\ell\ell'}\,\delta_{mm'}$$

assuming $|\ell_1-\ell_2| \le \ell \le \ell_1+\ell_2$

$$\sum_{\substack{|\ell_1-\ell_2|\le \ell\le \ell_1+\ell_2\\ m}} \begin{bmatrix}\ell_1 & \ell_2 & \ell\\ m_1 & m_2 & m\end{bmatrix}^{z_\ell} \begin{bmatrix}\ell_1 & \ell_2 & \ell\\ m_1' & m_2' & m\end{bmatrix}^{z_\ell^{-1}} = \delta_{m_1 m_1'}\,\delta_{m_2 m_2'}$$

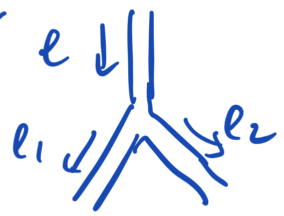
Choose $z_\ell = 1$ (Racah- Fock normalization)

Graphical calculus

$F\Bigg($  $\Bigg) = \varphi_\ell : V_\ell \to V_{\ell_1}\otimes V_{\ell_2}$

where φ_e is given by $(*)$ (assume $z_e=1$)

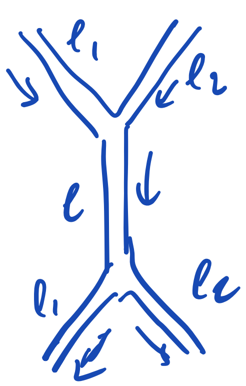
Similarly

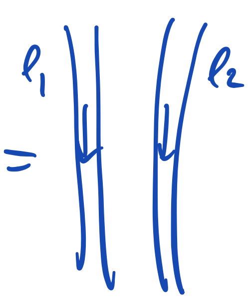
$F\Bigg($ 

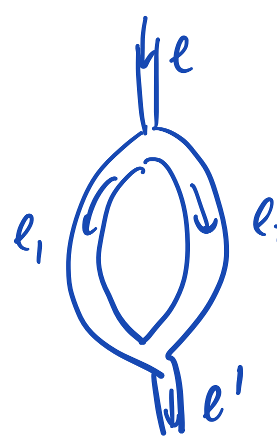
$\Bigg) = \pi_e : V_{e_1}\otimes V_{e_2} \to V_e$

where π_e is given by $(**)$ (assume $z_e=1$)

Because of relations (i) and (ii) we can impose the following relations on colored tangles and they will be consistent with framed Reidemeister moves and with the functor F.

$\sum_e$ 

$=$ 

, 

$=\delta_{ee'}$ 

Quantum Weyl group

Denote by τ the antiinvolution of the algebra $U_h sl_2$.

$$\tau(E) = Fe^{\frac{hH}{2}}, \quad \tau(F) = e^{-\frac{hH}{2}}E, \quad \tau(H) = H$$

Lemma τ is an automorphism of the coalgebra structure.

Proof. It is clear that $\Delta(\tau(H)) = (\tau\otimes\tau)\Delta(H)$.

$$\Delta(\tau(E)) = \Delta(Fe^{\frac{hH}{2}}) = (F\otimes 1 + e^{-\frac{hH}{2}}\otimes F)e^{\frac{hH}{2}}\otimes e^{\frac{hH}{2}}$$

$$= Fe^{\frac{hH}{2}}\otimes e^{\frac{hH}{2}} + 1\otimes Fe^{\frac{hH}{2}} = (\tau\otimes\tau)(\Delta E)$$

Similarly $\Delta(\tau(F)) = (\tau\otimes\tau)(\Delta F)$ ∎

We also have

Proposition

$$(\tau\otimes\tau)(R) = R_{21}$$

<u>Proof.</u> Let $R_0 = \exp\left(\frac{h}{2} H \otimes H\right)$,

$$(\tau \otimes \tau)(R) = \sum_{n \geq 0} \frac{\tau(E)^n \otimes \tau(F)^n}{(n)!} R_0$$

we have identities:

$$\tau(E)^n = \left(F e^{\frac{hH}{2}}\right)^n = F^n e^{\frac{h}{2} nH + hn(n-1)}$$

$$\tau(F)^n = \left(e^{-\frac{hH}{2}} E\right)^n = e^{-\frac{hH}{2} n} E^n e^{-hn(n-1)}$$

we also have the following identity

$$(F^n \otimes E^n) R_0 = (F^n \otimes 1) e^{\frac{h}{4} H \otimes (H-2n)} 1 \otimes E^n$$

$$= e^{\frac{h}{4} H \otimes H} F^n e^{-\frac{h}{2} nH} \otimes e^{\frac{hn}{2} H} E^n$$

Combining these identities we have

$$(\tau \otimes \tau)(R) = \exp\left(\frac{h}{4} H \otimes H\right) \sum_{n \geq 0} \frac{F^n \otimes E^n}{(n)!} =$$

$$= R_{21}$$

∎

Recall that the antipode acts on generators as

$$S(E) = -Ee^{-\frac{\lambda H}{2}}, \quad S(F) = -e^{\frac{\lambda H}{2}}F, \quad S(H) = -H$$

Define $\theta = S \circ \tau$. This is an automorphism of algebra $U_h sl_2$ and an anti-automorphism of the coalgebra structure.

It acts on generators as

$$\theta(E) = -F, \quad \theta(F) = -E, \quad \theta H = -H$$

and, because $(S \otimes S)(R) = R$,

$$(\theta \otimes \theta)(R) = R_{21}$$

Now we want to extend $U_h sl_2$ by adding an element w to it such that

$$\theta(a) = waw^{-1}$$

i.e. in the extended algebra θ becomes an inner automorphism.

Because $(\theta \otimes \theta)\Delta(a) = \Delta^{op}(\theta(a))$

the element

$$C = (w^{-1} \otimes w^{-1})\Delta w$$

should have the property

$$\Delta^{op}(a) = C\, \Delta(a)\, C^{-1}$$

Also

$$(\Delta \otimes id)C = (\Delta(w)^{-1} \otimes w^{-1})\Delta^{(2)} w =$$

$$= (C^{-1} \otimes 1)(w^{-1} \otimes w^{-1} \otimes w^{-1})\Delta^{(2)} w$$

$$(id \otimes \Delta)C = (1 \otimes C^{-1})(w^{-1} \otimes w^{-1} \otimes w^{-1})\Delta^{(2)}(w)$$

and therefore we should have

$$(C \otimes 1)(\Delta \otimes id)C = (1 \otimes C)(id \otimes \Delta)C$$

It is easy to check that $C = R$ satisfies this property.