Lecture 3

Last time:

- categories of framed tangles, framed diagrams
- framed diagrams and framed tangles colored by a ribbon category $\mathcal{C}$
- the evaluation functor.

Quantum sl_2

For the double construction see notes from the last semester. Quantum sl_2 is a quotient algebra of the Drinfeld double of its quantum Borel subalgebra.

- $U_h b_+$ over $\mathbb{C}[[h]]$:
  - generated by H and X with defining relations

    $$[H, E] = 2E$$

    complete over $\mathbb{C}[[h]]$, i.e. as a vector space is the space of formal power series $\sum^{\infty} c_n(H,E) h^n$

$n=0$

where $c_n(H,E)$ is a polynomial in H & X.

- it is a Hopf algebra with the comultiplication Δ acting on generators as

$$\Delta H = H\otimes 1 + 1\otimes H, \quad \Delta E = E\otimes e^{\frac{hH}{2}} + 1\otimes E$$

- Its restricted dual Hopf algebra $(U_h b_+)^\vee$ is generated by $H^\vee, E^\vee$ with defining relations

$$[H^\vee, E^\vee] = -\frac{h}{2}E^\vee$$

and with the comultiplication

$$\Delta H^\vee = H^\vee\otimes 1 + 1\otimes H^\vee, \quad \Delta E^\vee = E^\vee\otimes e^{-2H^\vee} + 1\otimes E^\vee$$

as a vector space it is $Pol(X)[[H^\vee, h]]$

The pairing with $U_h b_+$:

$$\langle (H^\vee)^n (E^\vee)^m, H^{n'} E^{m'}\rangle = \delta_{nn'}\delta_{mm'}$$

$n!\,(m)!$,

$$(m)! = \frac{1-e^{-hm}}{1-e^{-h}}\cdots\frac{1-e^{-h}}{1-e^{-h}},$$

- The double of $U_h b_+$, based on the dual pair of Hopf algebras $U_h b_+$, $U_h b_+^\vee$ is the Hopf algebra generated by $H, X, H^\vee, X^\vee$. As a vector space, it is the space of polynomials in $H, X, X^\vee$ complete over formal power series in $H^\vee, h$ with defining relations

$$[H,H^\vee]=0,\ [H,E]=2E,\ [H^\vee,E]=\frac{h}{2}E,$$

$$[H,E^\vee]=-2E^\vee,\ [H^\vee,E^\vee]=-\frac{h}{2}E^\vee$$

$$[E,E^\vee]=e^{\frac{hH}{2}}-e^{-2H^\vee}$$

The comultiplication acts on generators as

$$\Delta H=H\otimes 1+1\otimes H,\quad \Delta H^\vee=H^\vee\otimes 1+1\otimes H^\vee,$$

$$\Delta E=E\otimes e^{\frac{hH}{2}}+1\otimes E,\quad \Delta E^\vee=E^\vee\otimes 1+e^{-2H^\vee}\otimes E^\vee$$

This Hopf algebra is quasitriangular with the universal R-matrix

$$R=\sum_{n,m\geq 0}\frac{1}{n!\,(m)!}H^nE^m\otimes(H^\vee)^n(E^\vee)^m$$

$$= \exp(H \otimes H^\vee)\, f(E \otimes E^\vee; e^{-h}) \subset \mathcal{D}(U_h b_+)^{\hat{\otimes}}$$

$$f(z,q) = \sum_{m \geq 0} \frac{(1-q)^m z^m}{(1-q)\cdots(1-q^m)} = \prod_{n \geq 0} \left(1 - (1-q) z q^n\right)^{-1}$$

Here $\hat{\otimes}$ is the tensor product completed over $\mathbb{C}[[h, H^\vee \otimes 1, 1 \otimes H^\vee]]$.

- Define the algebra $U_h gl_2$ as the division algebra for $\mathcal{D}(U_h b_+)$ where $H^\vee$ and $E^\vee$ are divisible by h. Define

$$G = \frac{4H^\vee}{h}, \quad F = \frac{E^\vee}{e^{h/2} - e^{-h/2}}$$

i.e. $U_h gl_2$ is generated by H, G, E, F over $\mathbb{C}[[h]]$, complete over $\mathbb{C}[[h]]$ with

$$[H, G] = 0, \quad [H, E] = 2E, \quad [H, F] = -2F$$

$$[G, E] = 2E, \quad [G, F] = -2F, \quad [E, F] = \frac{e^{\frac{hH}{2}} - e^{-\frac{hG}{2}}}{e^{h/2} - e^{-h/2}},$$

with

$$\Delta H = H\otimes 1 + 1\otimes H, \quad \Delta G = G\otimes 1 + 1\otimes G,$$

$$\Delta E = E\otimes e^{\frac{hH}{2}} + 1\otimes E, \quad \Delta F = F\otimes 1 + e^{-\frac{hH}{2}}\otimes F,$$

and with

$$R = \exp\left(\frac{h}{4} H\otimes G\right) \sum_{n\geq 0} \frac{(e^{h/2} - e^{-h/2})^n}{(n)!} E^n \otimes F^n$$

- Finally, it is clear that $H-G$ generates a Hopf ideal (just as it generates a Lie ideal in gl_2). The quotient algebra over this ideal is $U_h sl_2$. it is generated by H, E, F with defining relations

$$[H,E] = 2E, \quad [H,F] = -2F$$

$$[E,F] = \frac{sh(\frac{h}{2}H)}{sh(\frac{h}{2})}$$

with the comultiplication

$$\Delta H = H\otimes 1 + 1\otimes H, \quad \Delta E = E\otimes e^{\frac{hH}{2}} + 1\otimes E, \quad \Delta F = F\otimes 1 + e^{-\frac{hH}{2}}\otimes F$$

with the antipode

$$S(H) = -H, \quad S(E) = -E e^{-\frac{hH}{2}}, \quad S(F) = -e^{\frac{hH}{2}} F$$

and it is quasitriangular with

$$R = \exp\left(\frac{h}{4} H \otimes H\right) \sum_{n \geq 0} \frac{(e^{h/2} - e^{-h/2})^n}{(n)!} E^n \otimes F^n$$

Important property:

<u>Thm</u>. 1) $U_h sl_2 \simeq U sl_2 [[h]]$ as an algebra (not as a Hopf algebra)

2) $Z(U_h sl_2)$ is generated by

$$c_2 = \left(\frac{sh\left(\frac{h}{4}(H+1)\right)}{sh\left(\frac{h}{2}\right)} \right)^2 + FE$$

Finite dimensional representations of $U_h sl_2$

Let $U_h b_+$ be the Borel subalgebra generated by H and E. Define $\mathcal{L}_\lambda$, $\lambda \in \mathbb{C}$ as the one dimensional representation of $U_h b_+$ with

$$H \mathcal{L}_\lambda = 2\lambda \mathcal{L}_\lambda, \quad X \mathcal{L}_\lambda = 0$$

Define the Verma module M_λ over $U_h sl_2$ as the induced representation

$$M_\lambda = \mathrm{Ind}_{U_h b_+}^{U_h sl_2} \mathcal{L}_\lambda$$

As a vector space

$$M_\lambda = U_h sl_2 \otimes_{U_h b_+} \mathcal{L}_\lambda \simeq U_h n_+ \otimes_{\mathbb{C}[[h]]} \mathcal{L}_\lambda$$

where $U_h n_+$ is the subalgebra generated by F. Choose a basis e_λ^λ in $\mathcal{L}_\lambda$. This gives a natural basis in M_λ

$$M_\lambda = \bigoplus_{n \geq 0} \mathbb{C} e_{\lambda-n}^\lambda$$

where $e_{\lambda-n}^\lambda = F^n e_\lambda^\lambda$.

Let us compute the action of generators on this basis. We immediately have

$$Y e^{\lambda}_{\lambda-n} = e^{\lambda}_{\lambda-n-1}$$

$$H e^{\lambda}_{\lambda-n} = H F^{n} e^{\lambda}_{\lambda} = F^{n}(H-2n) e^{\lambda}_{\lambda} = 2(\lambda-n) e^{\lambda}_{\lambda-n}$$

$$E e^{\lambda}_{\lambda-n} = E F^{n} e^{\lambda}_{\lambda} = \Bigg(\frac{sh(\frac{h}{2}H)}{sh(h/2)} F^{n-1} + \dots$$

$$+ F^{k} \frac{sh(\frac{h}{2}H)}{sh(\frac{h}{2})} F^{n-k-1} + \dots + F^{n-1} \frac{sh(\frac{h}{2}H)}{sh(\frac{h}{2})} \Bigg) e^{\lambda}_{\lambda}$$

$$= \frac{1}{sh(\frac{h}{2})} \Big(sh\big(\tfrac{h}{2}(2\lambda-2n+2)\big) + sh\big(\tfrac{h}{2}(\lambda-2n+4)\big) + \dots +$$

$$+ sh\big(\tfrac{h}{2}\lambda\big)\Big) e^{\lambda}_{\lambda-n+1}, \quad n \geq 1$$

i.e.

$$E e^{\lambda}_{\lambda-n} = \frac{sh\big(\frac{h}{2}(\lambda-n+1)\big)\, sh\big(\frac{h}{2}n\big)}{sh(h/2)^{2}} e^{\lambda}_{\lambda-n+1}, \quad n \geq 1$$

and, by definition of e^{λ}_{λ}, $E e^{\lambda}_{\lambda} = 0$.

The following is clear

Thm. M_λ is irreducible for generic complex λ (when $\lambda \neq 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$)

Indeed, it is easy to show that M_λ does not subrepresentations, except itself.

The action of H, E, F can be described by the diagram

$$\begin{tikzcd} \lambda \arrow[r, bend left, "F"] & \lambda-1 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & \lambda-2 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & \lambda-3 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & \lambda-4 \arrow[l, bend left, "E"] & \cdots \cdots \end{tikzcd}$$

Here vertices denote weight subspaces on which H acts diagonally.

However, when $\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ there is an extra singular vector for E:

$$E\, e^{\lambda}_{-\lambda-1} = 0$$

As a consequence vectors $e^{\lambda}_{-\lambda-1}, e^{\lambda}_{-\lambda-2}, \dots$ form a subrepresentation $M_{-\lambda-1} \subset M_{\lambda}$.

The quotient representation

$$V_{\lambda} = M_{\lambda} / M_{-\lambda-1}$$

with the weight basis e^{λ}_{m}, $m = \lambda, \lambda-1, \lambda-2, \dots, -\lambda$ is the $2\lambda+1$ - dimensional irreducible $U_h sl_2$ - - module.

Similarly to sl_2 this are all irreducible finite dimensional representations: any other is isomorphic to one of V_{ℓ}'s.

Also similarly to sl_2 it is easy to show that any finite dimensional representation is isomorphic to a direct sum of irreducibles, i.e. for any finite dimensional $U_h sl_2$ - module

V we have

$$V \simeq \bigoplus_{e} V_e^{\oplus m_e(V)}$$

where $m_e(V)$ is the multiplicity of V_e in V.

Finally, it is easy to compute the action of the Casimir on the highest weight and therefore on the whole M_λ

$$c_2 M_\lambda = [\lambda + \tfrac{1}{2}]^2 M_\lambda$$

and therefore

$$c_2 V_e = [e + \tfrac{1}{2}]^2 V_e$$

Here

$$[e] = \frac{sh(\frac{he}{2})}{sh(\frac{h}{2})}$$

<u>The R-matrix in $V_{e_1} \otimes V_{e_2}$</u>

Now we can compute the action of R on $V_{e_1} \otimes V_{e_2}$. Recall the action of generators of generators of $U_h sl_2$ on a weight basis in V_e :

$$H e_m = 2m\, e_m, \quad F e_m = e_{m-1},$$

$$E e_m = [l+m+1][l-m]\, e_{m+1},$$

For this action

$$E^n e_m = [l+m+1][l+m+2]\cdots[l+m+n]\cdot$$
$$\cdot [l-m][l-m-1]\cdots[l-m-n+1]\, e_{m+n}$$

$$F^n e_m = e_{m-n}$$

Let $(R)^{n_1 n_2}_{m_1 m_2}$ be matrix elements of R in the monomial basis $e_{m_1} \otimes e_{m_2}$:

$$R\, e_{m_1} \otimes e_{m_2} = \sum_{n_1, n_2} (R)^{n_1 n_2}_{m_1, m_2}\, e_{n_1} \otimes e_{n_2}$$

From the explicit formula for R we have:

$$R\, e_{m_1} \otimes e_{m_2} = \sum_{n \geq 0} \frac{e^{h\frac{n(n-1)}{4}}}{[n]!} \frac{[l+m_1+n]!\,[l-m_1]!}{[l+m_1]!\,[l-m_1-n]!} \left(e^{\frac{h}{2}} - e^{-\frac{h}{2}}\right)^n$$
$$e^{h(m_1+n)(m_2-n)}\, e_{m_1+n} \otimes e_{m_2-n}$$

here we used the identity

$$(n) = \frac{1-e^{-hn}}{1-e^{-h}} = e^{-\frac{h(n-1)}{2}}[n] \,, \quad (n)! = e^{-\frac{h}{2}(1+\cdots+(n-1))}[n]!$$

$$= e^{-\frac{h}{4}n(n-1)}[n]!$$

In particular when $l=\frac{1}{2}$,

$$R\, e_{\frac{1}{2}} \otimes e_{\frac{1}{2}} = e^{\frac{h}{4}}\, e_{\frac{1}{2}} \otimes e_{\frac{1}{2}}$$

$$R\, e_{\frac{1}{2}} \otimes e_{-\frac{1}{2}} = e^{-h/4}\, e_{\frac{1}{2}} \otimes e_{-\frac{1}{2}}$$

$$R\, e_{-\frac{1}{2}} \otimes e_{\frac{1}{2}} = e^{-\frac{h}{4}}\, e_{-\frac{1}{2}} \otimes e_{\frac{1}{2}} + (e^{h/2}-e^{-h/2})e^{-h/4}\, e_{\frac{1}{2}} \otimes e_{\frac{1}{2}}$$

$$R\, e_{-\frac{1}{2}} \otimes e_{-\frac{1}{2}} = e^{+h/4}\, e_{-\frac{1}{2}} \otimes e_{-\frac{1}{2}}$$

in the basis $\rightarrow \begin{pmatrix} e_{\frac{1}{2}} \otimes e_{\frac{1}{2}} \\ e_{\frac{1}{2}} \otimes e_{-\frac{1}{2}} \\ \hline e_{-\frac{1}{2}} \otimes e_{\frac{1}{2}} \\ e_{-\frac{1}{2}} \otimes e_{-\frac{1}{2}} \end{pmatrix}$

we have

$$R = q^{\frac{1}{2}} \left(\begin{array}{cc|cc} q & & & \\ & 1 & & \\ \hline & q-q^{-1} & 1 & \\ & & & q \end{array}\right),$$

This R-matrix gives Jones polynomial.

The decomposition of the tensor product

Let V_{ℓ_1} and V_{ℓ_2} be two finite dimensional $U_h sl_2$-modules. Their tensor product is finite dimensional and therefore is isomorphic to a direct sum of V_ℓ's. It turns out that

$$(*)\quad V_{\ell_1} \otimes V_{\ell_2} \simeq V_{\ell_1+\ell_2} \oplus V_{\ell_1+\ell_2-1} \oplus \cdots \oplus V_{|\ell_1-\ell_2|}$$

For example, the irreducible submodule with the highest weight $\ell_1+\ell_2$ is easy to construct:

- $e_{\ell_1}^{\ell_1} \otimes e_{\ell_2}^{\ell_2}$ is a highest weight vector (of weight $\ell_1+\ell_2$) in this tensor product, i.e. $H(e_{l_1}^{l_1}\otimes e_{l_2}^{l_2}) = (l_1+l_2)\, e_{l_1}^{l_1}\otimes e_{l_2}^{l_2}$ and any other highest weight vector is its multiple.

- The action of F on this vector generate the subrepresentation $V_{l_1+l_2} \subset V_{l_1}\otimes V_{l_2}$. It is easy to check that $F^{l_1+l_2+1}(e_{l_1}^{l_1}\otimes e_{l_2}^{l_2}) = 0$.

- The weight subspace of weight l_1+l_2-2 in this tensor product is two dimensional with the monomial basis $e_{l_1-2}^{l_1}\otimes e_{l_2}^{l_2}$ and $e_{l_1}^{l_1}\otimes e_{l_2-2}^{l_2}$. Vector

$$F(e_{l_1}^{l_1}\otimes e_{l_2}^{l_2}) = e_{l_1-2}^{l_1}\otimes e_{l_2}^{l_2} + e^{-hl_1} e_{l_1}^{l_1}\otimes e_{l_2-2}^{l_2}$$

is in $V_{l_2+l_2}^{l_1+l_2} \subset V_{l_1}\otimes V_{l_2}$. We can choose the second linearly independent vector in this weight subspace as a

highest weight vector of weight l_1+l_2-2.
The equation

$$E\left(a\, e^{l_1}_{l_1-2}\otimes e^{l_2}_{l_2} + b\, e^{l_1}_{l_1}\otimes e^{l_2}_{l_2-2}\right)=0$$

Fixes a and b uniquely up to a common multiple.

HW. Find $a:b$

Continuing this process one obtains an explicit decomposition of the tensor product into irreducible components.

Theorem [KR] Embeddings $V^l \hookrightarrow V^{l_1}\otimes V^{l_2}$ in the decomposition $(*)$ in terms of weight basis e^l_m

$$He^l_m = 2m e^l_m, \qquad Fe^l_m = e^l_{m-1}, \quad Fe^l_{-l}=0$$

$$E e^l_m = \frac{sh\left(\frac{h}{2}(l+m+1)\right) sh\left(\frac{h}{2}(l-m)\right)}{sh\left(\frac{h}{2}\right)^2}\, e^l_{m+1}$$

are given by formulae:

$$e^{l}_{m} = \sum_{m_1+m_2=m} \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}^{z_l} e^{l_1}_{m_1} \otimes e^{l_2}_{m_2}$$

where

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix} = z_l (-1)^{j_1-m_1} \frac{\rho(x)^{\frac{1}{2}}}{d_n} \, {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{x+1}, q^{-x} \\ aq, q^{-N} \end{matrix} ; q, q\right)$$

$$n = j-m, \quad a = q^{j_2-j+m_1}, \quad b = q^{j_1-j_2+m}, \quad x = j_1-m_1$$

$$N = j+j_2-m_1, \qquad q = e^{h}$$

$\rho(x)$, d_n see [KR].

$${}_{p+1}\varphi_p\left(\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix} ; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_{p+1}; q)_k}{(b_1, \dots, b_p; q)_k} z^k ,$$

$$(a;q)_k = \prod_{i=0}^{k-1} (1-aq^i) ,$$

$$(a_1, \dots, a_p; q)_k = \prod_{j=1}^{p} (a_j; q)_k ,$$

and z_l are arbitrary nonzero scalars.

<u>Remark</u>. The decomposition $(*)$ means that $\mathrm{Hom}(V_\ell, V_{\ell_1} \otimes V_{\ell_2})$ are 1-dimensional for $\ell_1+\ell_2 \geq \ell \geq |\ell_1-\ell_2|$ and zero otherwise. Different choices of z_ℓ correspond to different choices of the basis in $\mathrm{Hom}(V_\ell, V_{\ell_1} \otimes V_{\ell_2})$.