Ribbon categories

- Monoidal category
- Braided category
- Ribbon category

Ex.

- Category of modules over ribbon Hopf algebras
- $U_h sl_2$ as a ribbon Hopf algebra
- Ribbon tangles colored by a ribbon category $\mathcal{T}(\mathcal{C})$, zipping unzipping

Ribbon categories

Recall some basic definitions

1) A category is a collection of objects, morphisms between objects, and composition rules between morphisms. All categories here will be small (objects form a set and morphisms between objects form sets).

If $\mathcal{C}$ is a category, we denote $Ob(\mathcal{C})$ the set of objects. If $A, B \in Ob(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}(A, B)$$

is the set of morphisms from A to B.

Composition of morphisms is a map

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$$

$$(f, g) \mapsto gf$$

It is required to be associative:

$$h(gf) = (hg)f$$

for each f, g as above and $h \in \mathrm{Hom}_{\mathcal{C}}(C,D)$

Also, it is required that each $\mathrm{Hom}_{\mathcal{C}}(A,A)$ has an identity morphism $id_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$, and for each $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$

$$f \cdot id_A = id_B \cdot f$$

2) A monoidal category is a category with additional operation a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \,,$$

the "tensor product", together with an identity and an associator for this product.

More precisely, for each $A, B, C \in Ob(\mathcal{C})$ we have a functorial isomorphism

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

such that the <u>pentagon identity</u> holds:

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[rr, "id \otimes a_{B,C,D}"] & & A \otimes ((B \otimes C) \otimes D) \\
(A \otimes B) \otimes (C \otimes D) \arrow[u, "a_{A,B,C \otimes D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "a_{A,B \otimes C,D}"'] \\
& ((A \otimes B) \otimes C) \otimes D \arrow[ul, "a_{A \otimes B,C,D}"] \arrow[ur, "a_{A,B,C}"'] &
\end{tikzcd}$$

The unit for the tensor product is a unit object $\mathbb{1} \in Ob(\mathcal{C})$ with functorial isomorphism $l_A : \mathbb{1} \otimes A \xrightarrow{\sim} A$, $r_A : A \otimes \mathbb{1} \xrightarrow{\sim} A$, $r_{\mathbb{1}} = l_{\mathbb{1}} = u : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$

which satisfy natural axioms (see notes for the Fall course and [...] for details)

Functoriality of isomorphisms means that a is an isomorphism of functors

$$\begin{tikzcd}
 & \mathcal{C}\times\mathcal{C} \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow a"] & \\
\mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[ur, "\otimes\times id"] \arrow[dr, "id\times\otimes"'] & {} & \mathcal{C} \\
 & \mathcal{C}\times\mathcal{C} \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

Define functors $L_{\mathbb{1}}: \mathcal{C}\to\mathcal{C}$, $A\mapsto \mathbb{1}\otimes A$ and $R_{\mathbb{1}}: \mathcal{C}\to\mathcal{C}$, $A\mapsto A\otimes\mathbb{1}$, then systems of isomorphisms $\ell=\{\ell_A\}$ and $r=\{r_A\}$ define isomorphisms of functors:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "R_{\mathbb{1}}"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow r"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "L_{\mathbb{1}}"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \ell"] & \mathcal{C}
\end{tikzcd}$$

Again, for details see [ ].

In this course all monoidal categories will be strict, i.e.

$$(A\otimes B)\otimes C = A\otimes(B\otimes C)$$

and $a_{A,B,C}$. For the precise meaning of this see the Fall course and [...] for details.

3) A monoidal category is braided if for each pair $A, B \in Ob(\mathcal{C})$ an functorial isomorphism

$$c_{A,B} : A\otimes B \xrightarrow{\sim} B\otimes A$$

is given and if these isomorphisms satisfy the hexagon axioms, which is for strict monoidal categories is a collection of identities

$$c_{A\otimes B, C} =$$

$$c_{A, B\otimes C} =$$

Again, for details see, for example [ ].

4) A *right dual* to an object $A \in Ob(\mathcal{C})$ in a monoidal category is a triple

$$(A^*,\ e_A : A\otimes A^* \to \mathbb{1},\ i_A : \mathbb{1} \to A\otimes A^*)$$

where A^* is a dual object to A and e_A, i_A are evaluation and injection morphisms respectively. They should satisfy identities

$$(i_A \otimes id_A)(id_A \otimes e_V) = id_A$$

$$(id_{A^*} \otimes i_A)(e_V \otimes id_{A^*}) = id_{A^*}$$

Right dual to an object is unique, up to an isomorphism.

If every object in $\mathcal{C}$ has a right dual the category is called *rigid monoidal* category.

If ${}^*A$ is an object, such that $({}^*A)^* = A$ for some right dual to ${}^*A$, the object ${}^*A$ is called left dual.

5) Finally, a rigid braided monoidal category is called a ribbon category if for each object $A \in Ob(\mathcal{A})$ there exists a functorial isomorphism $v_A : A \to A$ such that

$$v_{\mathbb{1}} = id_{\mathbb{1}}$$

$$v_{A^*} = (v_A)^*$$

$$v_{A\otimes B} = v_A \otimes v_B \cdot c_{XY}^{-1} c_{YX}^{-1}$$

Ribbon categories will be central to the first part of our course.

Now some examples:

1) <u>The category of vector spaces $\underline{\mathrm{Vect}}_{\mathbb{C}}$ over $\mathbb{C}$.</u>

Objects = complex finite dimensional vector spaces.

$\mathrm{Hom}_{\underline{\mathrm{Vect}}}(A, B) = \mathbb{C}$ linear maps from A to B

$\mathrm{id}_A \in \mathrm{Hom}_{\underline{\mathrm{Vect}}}(A, A)$ is the identity map

The monoidal structure is given by the algebraic tensor product $A \otimes B = A \otimes_{\mathbb{C}} B$

$$A \otimes_{\mathbb{C}} B = A \times B \Big/ \Big\langle (x,y);\ (\lambda x, y) \sim (x, \lambda y) \sim \lambda(x,y) \text{ for any } \lambda \in \mathbb{C} \Big\rangle$$

Strictly speaking, $\underline{\mathrm{Vect}}_{\mathbb{C}}$ is not a strict monoidal category, but we will treat it as such, assuming the strictification [ ]

Any one-dimensional space can be regarded as a unit object. By a choice of basis we can identify each of them with $\mathbb{C}$. So our default choice of unit object in $\underline{\mathrm{Vect}}_{\mathbb{C}}$ is $\mathbb{C}$.

The space A^* of all $\mathbb{C}$-valued, $\mathbb{C}$-linear functions on A together with mappings

$$e_A : A^* \otimes A \to \mathbb{C}, \quad (\ell, x) \mapsto \ell(x)$$

and

$$\imath_A : \mathbb{C} \to A \otimes A^*, \quad 1 \to \sum_i e_i \otimes e^i$$

is the standard choice of the right dual. Here $\{e_i\}$ is a basis in A, $\{e^i\}$ is the dual basis in A^* ($e^i(e_j) = \delta^i_j$) and the element $\sum_i e_i \otimes e^i \in A \otimes A^*$ does not depend on the choice of $\{e_i\}$. It represents the unit operator $id_A : A \to A$ after natural identification $\mathrm{End}(A) \simeq A \otimes_{\mathbb{C}} A^*$.

The braiding in $\underline{\mathrm{Vect}}_{\mathbb{C}}$ is given by permutation maps

$$c_{AB} : A \otimes B \xrightarrow{\sim} B \otimes A, \quad x \otimes y \mapsto y \otimes x$$

The category $\underline{\text{Vect}}$ is symmetric

$$c_{AB}\, c_{BA} = id_{B\otimes A}$$

The ribbon structure is trivial $v_A = id_A$.

2) <u>The category of modules over a ribbon Hopf algebra</u>

<u>Def.</u> A Hopf algebra H is ribbon if

i) it is quasitriangular, i.e. $\exists R \in H\otimes H$ s.t.

$$(\Delta\otimes id)(R) = R$$

$$(id\otimes\Delta)(R) = R$$

$$\Delta^{op}(a) = R\,\Delta(a)\,R^{-1}$$

Here $\Delta^{op}(a) = \sigma(\Delta(a))$, $\sigma(a\otimes b) = b\otimes a$,

$R_{12} = R\otimes 1 \in H^{\otimes 3}$, $R_{23} = 1\otimes R \in H^{\otimes 3}$ etc.

ii) $\exists\, v \in H$, s.t. $\varepsilon(v) = 1$, $S(v) = v$

$$\Delta v = v\otimes v\ \ R\,\sigma(R)^{-1}$$

Let <u>H-mod</u> be the category of finite dimensional modules over H, i.e. pairs (π_V, V) where V is a vector space and $\pi_V : H \to End(V)$ is an algebra homomorphism.

Morphisms are H-linear maps:

$$Hom_H((\pi_V, V), (\pi_W, W)) = \{ f: V \to W,\ \pi_W(a) f = f \pi_V(a),\ \text{for any } a \in H \}$$

The identity morphism is the $\mathbb{C}$-linear identity map.

Tensor product is

$$(\pi_V, V) \otimes (\pi_W, W) = ((\pi_V \otimes \pi_W) \circ \Delta,\ V \otimes W)$$

An identity object is (ε, L), where L is any one dimensional space and ε is the counite in H.

The braiding is

$$c_{V,W} = P_{VW} (\pi_V \otimes \pi_W)(R) : V \otimes W \to W \otimes V$$

where $P_{VW}(x \otimes y) = y \otimes x$.

The right dual is

$$(V, \pi_V)^* = (V^*, \pi_V^* \circ S)$$

where $\pi_V^*(a) : V^* \to V^*$ is the linear dual to $\pi_V(a) : V \to V$ and S is the antipode in H.

The ribbon structure is given by

$$\{ v_V = \pi_V(v) \}_{V \in Ob(\underline{H\text{-mod}})}$$