Lecture 5

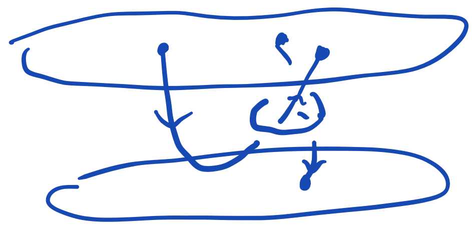
Last time:

- monoidal categories,
- monoidal functors,
- strict monoidal categories
- strictification
- examples: colored graphs, category of finite dimensional modules over bialgebras; dual bialgebras in the finite dimensional case.

- Now subject: tangles

• geometric tangles:

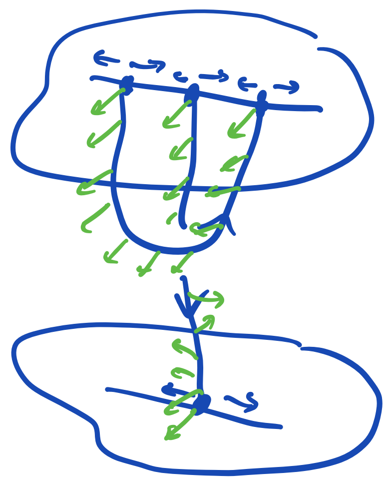
smooth oriented embeddings $I^{\sqcup k} \sqcup (S^1)^{\sqcup m} \hookrightarrow \mathbb{R}^2 \times [0,1]$, $I = [0,1]$



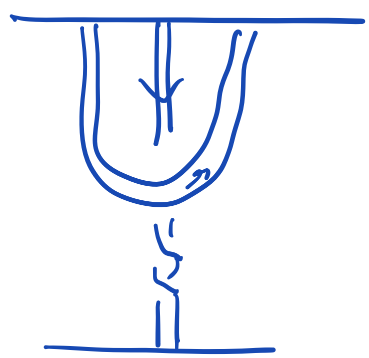
- ✓ • based tangles: relative (to the boundary) regular isotopy classes of tangles (fixed end-points)
- • standardized tangles = based tangles with endpoints on a line.
- ✓ • tangles = regular isotopy classes of standardized tangles.

Framed tangles

Similarly to framed braids and links we define framed tangles. We always assume the blackboard framing at the endpoints of a tangle:



. A tangle diagram is the regular isotopy class in $\mathbb{R} \times [0,1]$ of the projection of a generic representative of a framed tangle.



Here <u>generic</u> means the same as for links: there are only double singular points in the projection (undercrossings and overcrossings) and twists of framing, both are away from the boundary.

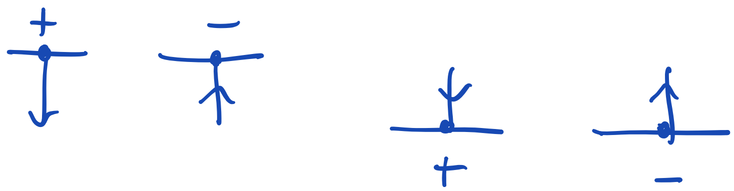
- As in case of links and braids framed Redemeister classes of tangle diagrams are in bijection with tangles.

(1) Based tangles form a category. $\mathcal{T}_{Based}$

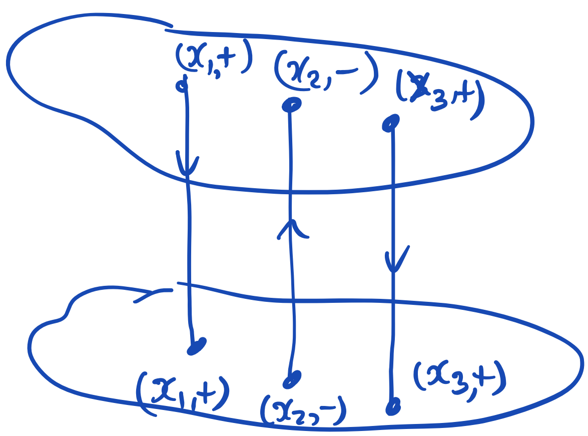
$Ob(\mathcal{T}_{based}) = \{(x_1,\varepsilon_1), \ldots, (x_n,\varepsilon_n) \mid \varepsilon_i = \pm, x_i \in \mathbb{R}\}$

$Hom_{\mathcal{T}_{based}}(\{x,\varepsilon\},\{y,\sigma\}) = \{$ based tangles with endpoints $\{x\}$ on $\mathbb{R}^2 \times \{1\}$, and $\{y\}$ on $\mathbb{R}^2 \times \{0\}\}$

The signs ε_i, σ_i determine the orientation of connected components of tangles near the boundary:



$Id_{\{x,\varepsilon\}}$ is the "untangle" with endpoints $\{x\} \in \mathbb{R}^2 \times \{0\}$ and $\{x\} \in \mathbb{R}^2 \times \{1\}$



② Tangles can be naturally organized into a monoidal category.

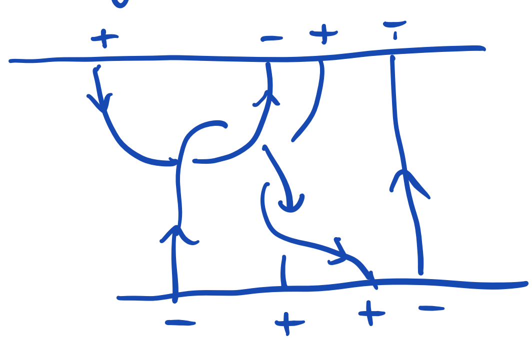
The category of tangles $\mathcal{T}$ has the following structure:

$$Ob(\mathcal{T}) = \{(\varepsilon_1, \dots, \varepsilon_n) \mid \varepsilon_i = \pm\}$$

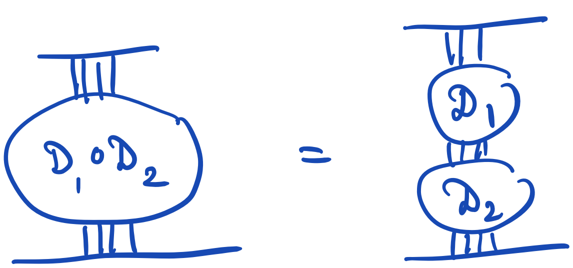
$\mathrm{Hom}_{\mathcal{T}}((\varepsilon),(\sigma)) = \{$ tangles with the orientation near the endpoints determined by (ε) and (σ) (as above) $\}$

We will think of tangles as Redemeister classes of tangle diagrams.

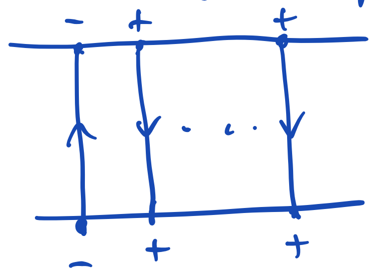
Here is an example of a diagram representing a tangle:



The composition of morphism is the gluing:

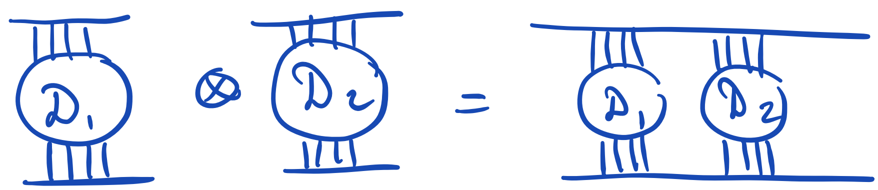


with the identity morphism



Monoidal structure is defined as:

$$(\varepsilon)\otimes(\sigma) = (\varepsilon_1,\dots,\varepsilon_n,\sigma_1,\dots,\sigma_m)$$



with the unit object being an empty sequence.

This is an example of a strict monoidal category.

Monoidal functors

Def. A monoidal functor F between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ with

- a mapping $F_{\mathbb{1}} : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$
- functorial isomorphisms $F_{A,B} : F(A \otimes B) \to F(A) \otimes F(B)$

s.t. the following diagrams are commutative

(i)
$$\begin{tikzcd}
F(A \otimes (B \otimes C)) \arrow[d, "F_{A,B\otimes C}"] & F((A \otimes B) \otimes C) \arrow[l, "F(a_{\mathcal{C}})"'] \arrow[d, "F_{A\otimes B, C}"] \\
F(A) \otimes_{\mathcal{D}} F(B \otimes C) \arrow[d, "\mathrm{id} \otimes F"] & F(A \otimes B) \otimes_{\mathcal{D}} F(C) \arrow[d, "F \otimes \mathrm{id}"] \\
F(A) \otimes_{\mathcal{D}} F(B) \otimes_{\mathcal{D}} F(C) & (F(A) \otimes F(B)) \otimes F(C) \arrow[l, "a_{\mathcal{D}}"']
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
F(\mathbb{1} \otimes B) \arrow[r, "F_{\mathbb{1},B}"] \arrow[d, "F(l_B)"'] & F(\mathbb{1}) \otimes_{\mathcal{D}} F(B) \\
F(B) & \mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(B) \arrow[l, "l_{F(B)}"'] \arrow[u, "F_{\mathbb{1}} \otimes \mathrm{id}"']
\end{tikzcd}$$

Here $l_B : \mathbb{1} \otimes B \xrightarrow{\sim} B$ is an autoequivalence and similar diagrams for $r_B : B \otimes \mathbb{1} \xrightarrow{\sim} B$

(iii)
$$\begin{tikzcd} F(A) \otimes_{\mathcal{D}} F(B) \arrow[rr, "F(f) \otimes_{\mathcal{D}} F(g)"] & & F(A') \otimes_{\mathcal{D}} F(B') \\ F(A \otimes B) \arrow[u, "F_{A,B}"] \arrow[rr, "F(f \otimes g)"] & & F(A' \otimes B') \arrow[u, "F_{A',B'}"] \end{tikzcd}$$

for any $f : A \to A'$, $g : B \to B'$

Example 1:

Let A, B be two bialgebras and $\varphi : A \to B$ be a homomorphism of bialgebras, i.e.

- $\varphi(ab) = \varphi(a)\varphi(b)$, $\varphi(1_A) = 1_B$
- $\Delta_B(\varphi(a)) = (\varphi \otimes \varphi)(\Delta_A(a))$
- $\varepsilon_B(\varphi(a)) = \varepsilon_A(a)$

Corresponding functor $F^{\varphi} : \underline{B\text{-mod}} \to \underline{A\text{-mod}}$

- $F^{\varphi}(V, \pi_V^B) = (V, \pi_V^B \circ \varphi)$

- if $f: V \to W$ is a linear map of B-modules, commuting with the B-action, i.e. a morphism in <u>B-mod</u>

$$F^{\varphi}(f) = f : V \to W$$

commutes with the A-action given by $\pi_V^B(\varphi(a))$, $a \in A$.

- $F^{\varphi}(V \otimes W, (\pi_V^B \otimes \pi_W^B)\circ\Delta) =$

$$= (V \otimes W, (\pi_V^B \otimes \pi_W^B)\circ\Delta\circ\varphi) =$$

$$= (V \otimes W, (\pi_V^B\circ\varphi \otimes \pi_W^B\circ\varphi)\circ\Delta) =$$

$$= F^{\varphi}(V, \pi_V^B) \otimes F^{\varphi}(W, \pi_W^B)$$

- $F^{\varphi}(L, \varepsilon^B) = (L, \varepsilon^B\circ\varphi) = (L, \varepsilon^A)$ for any 1-dimensional space L.

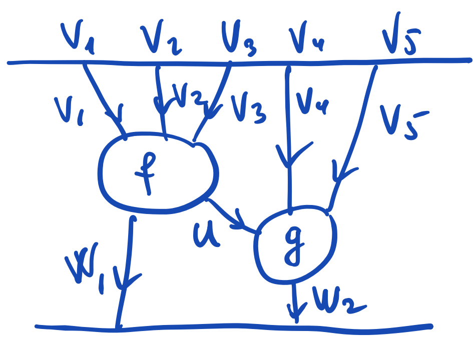
<u>Example 2</u>: Monoidal functor from the

category of planar graphs colored by vector spaces to vector spaces.

<u>Objects</u>: 

, $V_i \in \underline{Vect}_k$

<u>Morphisms</u> $Hom(\{W\}, \{V\})$ are



$f: W_1 \otimes U \to V_1 \otimes V_2 \otimes V_3$

$g: W_2 \to U \otimes V_4 \otimes V_5$

unit object = empty set.

The monoidal functor to the category of vector spaces:

$$F((V_1, .., V_n)) = V_1 \otimes \cdots \otimes V_n$$

$$F(\Gamma : (W_1, .., W_m) \to (V_1, .., V_n)) =$$

= The linear map $W_1 \otimes .. \otimes W_m \to V_1 \otimes .. \otimes V_n$ obtained by the composition of maps corresponding to vertices and id maps.

$F(\Gamma$ in the example above$) =$

$$= (f \otimes id_{V_4} \otimes id_{V_5}) \circ (id_{W_1} \otimes g)$$

$$F(\{V\} \otimes \{W\}) = V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m$$

$$F(\{V\}) \otimes F(\{W\}) = (V_1 \otimes \cdots \otimes V_n) \otimes (W_1 \otimes \cdots \otimes W_m)$$

$$F_{\{V\},\{W\}} : V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m \to$$

$$\to (V_1 \otimes \cdots \otimes V_n) \otimes (W_1 \otimes \cdots \otimes W_m)$$

is the natural isomorphism that we discussed above.

<u>Remark</u> Similarly we can construct a functor $F : \mathcal{T}(\mathcal{C}) \to \mathcal{C}$ where $\mathcal{T}(\mathcal{C})$ is the category of graphs colored by the monoidal category $\mathcal{C}$.

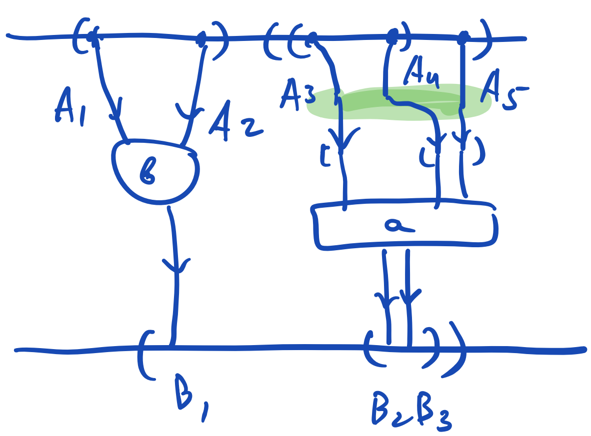
Example 3

1. Bracketed graphs colored by a monoidal category $\Gamma_B(\mathcal{C})$;

   <u>Objects</u>: bracketed sequences of objects in $\mathcal{C}$

   $((A_1, A_2), (A_3, A_4))$, etc.

   <u>Morphisms</u>: regular isotopy classes of directed colored graphs with associator vertices



$b : A_1 \otimes A_2 \to B_1$

$a : A_3 \otimes (A_4 \otimes A_5) \to$

$\to B_1 \otimes B_2$

...

(HW, fill in details)

monoidal functor $F : \Gamma_B(\mathcal{C}) \to \mathcal{C}$

$$F((A_1, \ldots, A_n)_B) = (A_1 \otimes A_2 \cdots \otimes A_3)_B$$

bracketed tensor product

$F(\Gamma)$ = the composition of maps corresponding to vertices and of associators.

Morphisms of monoidal functors.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

$$F_{A,B} : F(A \otimes B) \to F(A) \otimes F(B)$$
$$G_{A,B} : G(A \otimes B) \to G(A) \otimes G(B)$$

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_A"'] & G(B) \end{tikzcd}
\qquad
\begin{tikzcd} F(A \otimes B) \arrow[r, "\alpha_{A \otimes B}"] \arrow[d, "F_{AB}"] & G(A \otimes B) \arrow[d, "G_{A,B}"] \\ F(A) \otimes F(B) \arrow[r, "\alpha_A \otimes \alpha_B"] & G(A) \otimes G(B) \end{tikzcd}$$

Def. Two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent as monoidal categories if functors F & G in the def. of equivalence are monoidal and $F \circ G \sim \mathrm{id}$, $G \circ F \sim \mathrm{id}$ are monoidal equivalences.

Rigid monoidal categories

Rigidity of $\underline{\mathrm{Vect}}_k$:

For finite dimensional vector spaces there is an important notion of the dual vector space. If $V \in \underline{\mathrm{Vect}}_k$, $V^* \in \underline{\mathrm{Vect}}_k$ is the space of linear functions $\ell : V \to k$. If $f : V \to W$ is a linear map, $f^* : W^* \to V^*$ is the dual linear map, $f^*(\ell)(v) = \ell(fv)$.

For two composable morphisms f and g, $(fg)^* = g^* f^*$. For the tensor product $(V \otimes W)^* \simeq$ $\simeq W^* \otimes V^*$ with the corresponding property of morphisms. Thus, we have a contravariant functor

$$* : \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k$$

of taking duals.